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Disorder Solutions for Generalized Ising and Potts Models with Multispin Interaction

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Abstract

In this work, a special elementary transfer matrix is constructed for generalized Ising models and Potts models with the general form of a finite Hamiltonian with a multi-spin interaction in a space of arbitrary dimensionality, the Napierian logarithm of its maximum eigenvalue is equal to the free energy of the system. In some cases, it was possible to obtain an explicit form of the eigenvector corresponding to the largest eigenvalue of the elementary transfer matrix.
On this basis we obtained systems of nonlinear equations for the interaction coefficients of the Hamiltonian for finding the exact value of the free energy on a set of disorder solutions. Using the Levenberg-Marquardt method, the existence of nontrivial solutions of the resulting systems of equations for plane and three-dimensional Ising models was shown. In some special cases (the 2D Ising model, the interaction potential, including the interaction of the next nearest neighbors and quadruple interactions; the 3D model with a special Hamiltonian symmetric relative to the change of all spin signs, for which it is possible to reduce the system of equations to the system for a planar model) three parameters are written in explicit form. The domain of existence of these solutions is described.

Keywords: generalized Ising model, generalized Potts model, Hamiltonian, multi-spin interaction, transfer matrix, disorder solutions, statistical sum, free energy.

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Неупорядоченные решения обобщенных моделей Изинга и Поттса с мультиспиновым взаимодействием

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Аннотация

В работе построена специальная элементарная трансфер-матрица для обобщенных моделей Изинга и моделей Поттса с общим видом финитного гамильтониана с мультиспиновым взаимодействием в пространстве произвольной размерности, натуральный логарифм максимального собственного значения которой равен свободной энергии системы. В некоторых случаях удалось получить явный вид собственного вектора, отвечающего наибольшему собственному значению элементарной трансфер-матрицы. На основе этого выведены системы нелинейных уравнений на коэффициенты взаимодействия гамильтониана для нахождения точного значения свободной энергии на множестве неупорядоченных решений (disorder solutions). Методом Левенберга-Марквардта показано существование нетривиальных решений получающихся систем уравнений для плоских и трехмерных моделей Изинга. В некоторых частных случаях (2D модель Изинга, потенциал взаимодействия, включающий взаимодействие следующих ближайших соседей и четверные взаимодействия; 3D модель со специальным гамильтонианом, симметричным относительно перемены всех знаков спинов, для которой удается свести систему уравнений к системе для плоской модели) решения, зависящие от трех параметров, выписаны в явном виде. Описана область существования этих решений.

Ключевые слова: обобщенная модель Изинга, обобщенная модель Поттса, гамильтониан, мультиспиновое взаимодействие, трансфер-матрица, неупорядоченные решения (disorder solutions), статистическая сумма, свободная энергия.

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Introduction

The search for the exact values of free energy and other physical characteristics of various physical models attracted the attention of scientists all over the world over many years. The most brilliant achievement in this direction undoubtedly is the accurate solution of the two-dimensional Ising model without an external magnetic field, obtained by Onsager [1]. For the Ising and Potts models, few accurate solutions are known (with an analytical expression for the partition function or the free energy of the system) in the case when the interaction Hamiltonian includes a magnetic field, good reviews Wu F.Y. [2], Baxter [3]. Some examples are in Verhagen [4] for anisotropic models on a triangular lattice and Rujan [5] for the Ising and Potts models, respectively. These solutions belong to the class of so-called disorder solutions (Stephenson J. [6], Welberry T R and Galbraith R [7], Enting I.G [8]): these are solutions obtained on a certain subset in the parameter space of the physical system.

Different methods were used to obtain these solutions for different models: methods related to crystal growth (Welberry TR and Miller GH [9]), Markov processes (Verhagen [4], Rujan [5]), or the angular transfer matrix method (corner transfer-matrix) (Baxter [10]). The transfer-matrix apparatus is widely used in statistical physics [11–12]. In most cases, the task of calculating a partition function was compared with an equivalent one in another area, where the corresponding methods made it possible to solve the problem. A local criterion for obtaining such solutions is given in Jaekel and Maillard [13]. With its help, for the anisotropic Ising and Potts model with a magnetic field, a variety of values (disorder solutions) for the partition function for a chessboard type grid have been obtained. Publications [14–22] are also a devoted to the topics of disordered solutions.

In this paper, the author considers the generalized Ising and Potts models with a general view of the multi-spin interaction with boundary conditions with a shift (similar to screw ones), and the cyclic closure of the set of all points (in natural ordering); builds elemental transfer matrices for these models (for two-dimensional models, matrices of similar structure were used in [23]), writes out systems of equations for finding their maximum eigenvalues (and the exact form of eigenvectors corresponding to these maximum eigenvalues). The Napierian logarithm of the largest eigenvalue is the free energy of the system with parameters that satisfy the resulting system of equations. For a wide enough variety of models, exact solutions of these systems of equations are given in explicit form, depending on several parameters, as well as the range of permissible values for the solutions are obtained. In other cases, the Levenberg-Marquardt algorithm [24] showed the existence of nontrivial solutions of the written systems.

The high symmetry and repeatability of the components of the found eigenvectors, which disappear when the set of exact solutions is exceeded, is the reason for the search for phase transitions in the set neighborhood.

Disorder exact solutions will certainly help in the computational investigation of models. Comparison of the results of numerical simulation with exact results, at least on some set, will certainly raise the accuracy of numerical simulation.

Generalized Ising model

Analyze ν - dimensional grid

$\mathscr{L}_\nu = \{t = (t_1, t_2, ..., t_i, t_{i+1}, ..., t_\nu), t_i = 0,1,...,L_i, i = 1,2,...,\nu\}$, where

$(t_1, t_2, ..., L_i, t_{i+1}, ..., t_\nu) \equiv (t_1, t_2, ..., 0, t_{i+1}+1, ..., t_\nu)$, $i = 1,2,...,\nu-1$,

$$(L_1, L_2-1, ..., L_i-1, ..., L_\nu-1) \equiv (0, L_2, ..., L_i-1, ..., L_\nu-1) \equiv ... \equiv (0,0,...,L_\nu) \equiv (0,0,...,0) \quad (1)$$

Due to such a procedure of identifying points, the grid $\mathscr{L}_\nu$ has the size $L_1 \times L_2 \times ... \times L_\nu$, the total number of grid points $L = L_1 L_2 ... L_\nu$. Eo ipso on the $\mathscr{L}_\nu$ special boundary cyclic screw conditions (with a shift) are specified. Renumber all points of the grid $\mathscr{L}_\nu$:

$\tau_0 = (0,0,...,0)$, $\tau_1 = (1,0,...,0)$, $\tau_2 = (2,0,...,0)$, ...,

$$\tau_{L_1} = (L_1, 0, ..., 0) \equiv (0,1,0,...,0), \ \tau_{L_1+1} = (1,1,0,...,0), ..., \tau_L = (0,...,0) = \tau_0 \quad (2)$$

This numbering determines the natural cyclic round of all points (in the positive direction) and local (cyclic) ordering.

Assume that in the each point $t = (t_1, t_2, ..., t_\nu)$ there is a particle. The state of a particle is determined by the spin σ_t, which at each point of grid $t = (t_1, t_2, ..., t_\nu)$ can take two values: $\sigma_t \in X = \{+1, -1\}$.

Let us assume that $\Omega = \{ t^1, t^2, ..., t^p \}$- is some fixed finite subset (of a certain form) of points $\mathscr{L}_\nu$, we call it the carrier (or the carrier of the Hamiltonian), whose lowest point $t^{\min} = (0,...,0)$, the oldest $t^{i_{\max}} = (t_1^{i_{\max}}, t_2^{i_{\max}}, ..., t_\nu^{i_{\max}})$ (it does not mean all points $t^i, i = 0,1,2...,i_{\max}$, belong to |. For example, $\Omega = \{t = (t_1, t_2, ..., t_\nu) \in \mathscr{L}_\nu : t_i = 0,1, i = 1,2,...,\nu\}$ - unit ν -dimensional cube).

Hamiltonian of the model has the form

$$\mathscr{H}(\sigma) = -\sum_{i=0}^{L-1} \sum_{\{t^1, t^2, ..., t^s\} \subseteq \Omega_{\tau^i}} J_{t^1, t^2, ..., t^s} \sigma_{t^1} \sigma_{t^2} ... \sigma_{t^s} \quad (3)$$

where $\tau^i = (\tau_1^i, ..., \tau_\nu^i) \in \mathscr{L}_\nu$, $\Omega_{\tau^i} = \Omega + \tau^i$, $\{ t^1, t^2, ..., t^s \} \subseteq \Omega_{\tau^i}$ - is some nonempty subset Ω_{τ^i}, $J_{t^1, t^2, ..., t^s}$ - are corresponding translation-invariant coefficients of multi-spin interaction.

Such a notation allows formula (3) to describe any Hamiltonian with a finite support. Note that the same subset $\{ t^1, t^2, ..., t^s \} \subseteq \Omega_{\tau^i}$ can occur when different τ^i. In this case, when recording Hamiltonian (3) in standard form

$$\mathscr{H}(\sigma) = -\sum_{\{t^1, t^2, ..., t^s\}} \tilde{J}_{t^1, t^2, ..., t^s} \sigma_{t^1} \sigma_{t^2} ... \sigma_{t^s} \quad (4)$$

we get $\tilde{J}_{t^1, t^2, ..., t^s} = \sum_{\tau^i : \{t^1, t^2, ..., t^s\} \subseteq \Omega_{\tau_i}} J_{t^1, t^2, ..., t^s}$ (5)

We introduce the coefficients $K_{t^1, t^2, ..., t^s} = J_{t^1, t^2, ..., t^s} / (k_B T)$, where T - temperature, k_B - Boltzmann constant. Then the statistical sum of the model is written as

$$Z_L = \sum_{\{\sigma\}} \exp(-\mathscr{H}(\sigma) / (k_B T)) = \sum_{\{\sigma\}} \exp(\sum_{i=0}^{L-1} \sum_{\{t^1, t^2, ..., t^s\} \subseteq \Omega_{\tau^i}} K_{t^1, t^2, ..., t^s} \sigma_{t^1} \sigma_{t^2} ... \sigma_{t^s}) , \quad (6)$$

summation is performed over all states of spins.

We introduce an elementary transfer matrix $\mathscr{T} = \mathscr{T}_{p,r}$ of size $2^{i_{\max}} \times 2^{i_{\max}}$, nonzero elements of which $\mathscr{T}_{p,r}$ correspond to a pair of sets $\{(\sigma_{\tau^i}, \sigma_{\tau^{i+1}}, ..., \sigma_{\tau^{i+i_{\max}-1}}), (\sigma_{\tau^{i+1}}, \sigma_{\tau^{i+2}}, ..., \sigma_{\tau^{i+i_{\max}}})\}$, (7) at the same time (Fig. 1).

$$p = \sum_{k=0}^{i_{\max}-1} \frac{(1-\sigma_{\tau^{i+k}})}{2} 2^k, \ p = 0,1,2,...,2^{i_{\max}}-1, \quad (8)$$

$$r = \sum_{k=0}^{i_{\max}-1} \frac{(1-\sigma_{\tau^{i+1+k}})}{2} 2^k, \ r = 0,1,2,...,2^{i_{\max}}-1. \quad (9)$$

$$\mathscr{T}_{\{(\sigma_{\tau^0}, \sigma_{\tau^1}, ..., \sigma_{\tau^{i_{\max}-1}}), (\sigma_{\tau^1}, \sigma_{\tau^2}, ..., \sigma_{\tau^{i_{\max}}})\}} = \exp(\sum_{\{t^1, t^2, ..., t^s\} \subseteq \Omega_{\tau^0}} K_{t^1, t^2, ..., t^s} \sigma_{t^1} \sigma_{t^2} ... \sigma_{t^s}) = \mathscr{T}_{p,r}, \quad (10)$$

Then

$$Z_L = \sum_{\{\sigma_{\tau^0}, \sigma_{\tau^1}, ..., \sigma_{\tau^{L-1}}\}} T_{\{(\sigma_{\tau^0}, \sigma_{\tau^1}, ..., \sigma_{\tau^{i_{\max}-1}}), (\sigma_{\tau^1}, \sigma_{\tau^2}, ..., \sigma_{\tau^{i_{\max}}})\}} \cdot T_{\{(\sigma_{\tau^1}, \sigma_{\tau^2}, ..., \sigma_{\tau^{i_{\max}}}), (\sigma_{\tau^2}, \sigma_{\tau^3}, ..., \sigma_{\tau^{i_{\max}+1}})\}} \cdots T_{\{(\sigma_{\tau^{L-1}}, \sigma_{\tau^0}, ..., \sigma_{\tau^{i_{\max}-2}}), (\sigma_{\tau^0}, \sigma_{\tau^1}, ..., \sigma_{\tau^{i_{\max}-1}})\}} = Tr(\boldsymbol{T}^L) \quad (11)$$

Note that nonzero matrix elements of the matrix $\mathcal{T} = \mathcal{T}_{p,r}$ will be equal if the set $(\sigma_{\tau^0}, \sigma_{\tau^1}, \ldots, \sigma_{\tau^{i_{\max}}})$ includes the same subset $\sigma(\Omega_{\tau^0}) = \{\sigma_t, t \in \Omega_{\tau^0}\}$, with fixed spin values $\{\sigma_t, t \in \Omega_{\tau^0}\}$. Let

$$\tilde{\Omega}_{\tau^i} = \Omega_{\tau^i} \setminus \tau^{i+i_{\max}},\ i = 0,1,\ldots,L. \quad (12)$$

The eigenvector of the elementary transfer matrix $\mathcal{T} = \mathcal{T}_{p,r}$, corresponding to the highest eigenvalue F is sought in the form

$$\vec{b} = \{b_{(\sigma_{\tau^0}, \sigma_{\tau^1}, \ldots, \sigma_{\tau^{i_{\max}-1}})}, \sigma_{\tau^i} \in X, i = 0,1,\ldots,i_{\max}-1\} = \quad (13)$$

$$\{b_p, b_p > 0, p = \sum_{k=0}^{i_{\max}-1} \frac{(1-\sigma_{\tau^k})}{2} 2^k, p = 0,1,\ldots, 2^{i_{\max}-1} - 1\}$$

at the same time we consider that

$$b_{(\sigma_{\tau^0}, \sigma_{\tau^1}, \ldots, \sigma_{\tau^{i_{\max}-1}})} = \hat{b}(\sigma(\tilde{\Omega}_{\tau^0})), \quad (14)$$

that is, the components of the vector $\vec{b}$ will be the same if the set $(\sigma_{\tau^0}, \sigma_{\tau^1}, \ldots, \sigma_{\tau^{i_{\max}-1}})$ includes the same subset $\sigma(\tilde{\Omega}_{\tau^0})$ (total different components $\hat{b}(\sigma(\tilde{\Omega}_{\tau^0}))$ will be $2^{|\tilde{\Omega}_{\tau^0}|}$). Next will be shown the feasibility of this assumption. According to the Perron-Frobenius theorem [25], the maximum eigenvalue of a matrix with positive matrix elements (they will be positive for the transfer matrix to some degree; when multiplying the matrix by itself, it is easy to see the nature of the inevitable filling of the matrix with nonzero elements) all components of the vector $\vec{b}$ must be positive. In this case, we can write the system of equations for the eigenvector of the matrix $\mathcal{T} = \mathcal{T}_{p,r}$, assuming that the components of the eigenvector $\{\hat{b}(\sigma(\tilde{\Omega}_{\tau^0}))\}$ and the interaction coefficients $K_{t^1,t^2,\ldots,t^s}$ in the Hamiltonian are unknown:

$$b_{\{(\sigma_{\tau^0}, \sigma_{\tau^1}, \ldots, \sigma_{\tau^{i_{\max}-1}})\}} F = \quad (15)$$

$$\sum_{\sigma_{\tau^{i_{\max}}} \in X} \mathcal{T}_{\{(\sigma_{\tau^0}, \sigma_{\tau^1}, \ldots, \sigma_{\tau^{i_{\max}-1}}), (\sigma_{\tau^1}, \sigma_{\tau^2}, \ldots, \sigma_{\tau^{i_{\max}}})\}}$$

$$(\{K_{t^1,t^2,\ldots,t^s}\}) b_{\{(\sigma_{\tau^1}, \sigma_{\tau^2}, \ldots, \sigma_{\tau^{i_{\max}}})\}}$$

where F - higher eigenvalue of an elementary transfer matrix $\mathcal{T} = \mathcal{T}_{p,r}$. Taking into account (14) let's rewrite the system of equations (15) as

$$\hat{b}(\sigma(\tilde{\Omega}_{\tau^0})) F = \sum_{\sigma_{\tau^{i_{\max}}} \in X} \mathcal{T}_{\{(\sigma_{\tau^0}, \sigma_{\tau^1}, \ldots, \sigma_{\tau^{i_{\max}-1}}), (\sigma_{\tau^1}, \sigma_{\tau^2}, \ldots, \sigma_{\tau^{i_{\max}}})\}} \hat{b}(\sigma(\tilde{\Omega}_{\tau^1})), \quad (16)$$

where $\sigma_{\tau^{i_{\max}}}$ is included in the set $\sigma(\tilde{\Omega}_{\tau^1})$.

Or

$$b(\sigma(\Omega_{\tau^0})) F = \quad (17)$$

$$\sum_{\sigma_{\tau^{i_{\max}}} \in X} \exp\Big(\sum_{\{t^1,t^2,\ldots,t^s\} \subseteq \Omega_{\tau^0}} K_{t^1,t^2,\ldots,t^s} \sigma_{t^1} \sigma_{t^2} \ldots \sigma_{t^s}\Big) \hat{b}(\tilde{\Omega}_{\tau^1})$$

Entire $2^{|\tilde{\Omega}_{\tau^0}|}$ equations. Evaluate F in every equation and equating to the value F from the first equation, we get $2^{|\Omega_{\tau^0}|} - 1$ equation. The number of coefficients of multi-spin interactions $K_{t^1,t^2,\ldots,t^s}$ will be $2^{|\Omega_{\tau^0}|} - 1$ (empty set we remove). Given the fact that the various components $\hat{b}(\sigma(\tilde{\Omega}_{\tau^0}))$ will be $2^{|\tilde{\Omega}_{\tau^0}|}$, we get, that the solution of the system (17) is multiparameter, and it allows you to find not only free energy $f = \ln(F)$, but some other characteristics. Considering that free energy f does not depend on the size $\mathcal{L}_v$, then it remains the same as the size of the model tends to infinity. That is, we obtain free energy for a system of infinite size.

Generalized Potts model

Exactly the same reasoning and similar systems of equations are obtained if we consider the generalized Potts model with the Hamiltonian

$$\mathcal{H}_{Potts}(\sigma) = -\sum_{i=0}^{L-1} \sum_{\{t^1,t^2,\ldots,t^s\} \subseteq \Omega_{\tau^i}} \sum_{\{\mu_1,\mu_2,\ldots,\mu_s\} \in X^s} J_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}} \chi_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}(\sigma) \quad (18)$$

where $\tau^i = (\tau_1^i, \ldots, \tau_v^i)$, $\Omega_{\tau^i} = \Omega + \tau^i$, $\{t^1, t^2, \ldots, t^s\} \subseteq \Omega_{\tau^i}$ - some subset Ω_{τ^i}, $\{\propto_{t^1}, \propto_{t^2}, \ldots, \propto_{t^s}\}$ - some set of spins, $\mu_{t^i} \in X = \{1,2,\ldots,q\}$, at appropriate points $\{t^1, t^2, \ldots, t^s\} \subseteq \Omega_{\tau^i}$, $J_{\propto_{t^1},\propto_{t^2},\ldots,\propto_{t^s}}$ - corresponding translation-invariant coefficients of interspin interaction,

$$\chi_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}(\sigma) = \begin{cases} 1, if & \sigma_{t^i} = \mu_{t^i}, i = 1,\ldots,s \\ & 0, other\ cases \end{cases}$$

- characteristic function.

Such a notation allows you to describe an arbitrary Hamiltonian with a finite support with the formula (18). We introduce $K_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}} = J_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}} / (k_B T)$, where T - temperature, k_B - Boltzmann's constant. Then the statistical sum of the model is written as

$$Z_L = \sum_{\{\sigma\}} \exp(-\mathcal{H}_{Potts}(\sigma) / (k_B T)) =$$

$$\sum_{\{\sigma\}} \exp\Big(\sum_{i=0}^{L-1} \sum_{\{t^1,t^2,\ldots,t^s\} \subseteq \Omega_{\tau^i}} \sum_{\{\mu_1,\mu_2,\ldots,\mu_s\} \in X^s} K_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}} \chi_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}(\sigma)\Big)$$

where summation is performed over all states of spins.

We introduce a function

$$\mathcal{F}(\sigma_{\tau^i}, \sigma_{\tau^{i+1}}, \ldots, \sigma_{\tau^{i+i_{\max}}}) =$$

$$\exp\Big(\sum_{\{t^1,t^2,\ldots,t^s\} \subseteq \Omega_{\tau^i}} \sum_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}} K_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}} \chi_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}(\sigma_{\Omega_{\tau^i}})\Big)$$

where $\sigma_{\Omega_{\tau^i}} = \{\sigma_\tau, \tau \in \Omega_{\tau^i}\}$ the function $\chi_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}(\sigma_\Omega)$ depends not on all values of the spins in the points $\tau^i, \tau^{i+1}, \ldots, \tau^{i+i_{\max}}$, so we can write $\chi_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}(\sigma_{\Omega_{\tau^i}})$.

Then $Z_L = \sum_{\sigma} \prod_{i=0}^{L-1} \mathcal{F}(\sigma_{\tau^i}, \sigma_{\tau^{i+1}}, \ldots, \sigma_{\tau^{i+i_{\max}}})$

We introduce an elementary transfer matrix. $\mathcal{T} = \mathcal{T}_{p,r}$ with the size $q^{i_{\max}} \times q^{i_{\max}}$, nonzero elements of which $\mathcal{T}_{p,r}$ are numbered by pairs of sets $\{(\sigma_{\tau^i}, \sigma_{\tau^{i+1}}, \ldots, \sigma_{\tau^{i+i_{\max}-1}}), (\sigma_{\tau^{i+1}}, \sigma_{\tau^{i+2}}, \ldots, \sigma_{\tau^{i+i_{\max}}})\}$,

$$\mathcal{T}_{\{(\sigma_{\tau^0}, \sigma_{\tau^1}, \ldots, \sigma_{\tau^{i_{\max}-1}}), (\sigma_{\tau^1}, \sigma_{\tau^2}, \ldots, \sigma_{\tau^{i_{\max}}})\}} =$$

$$\exp\Big(\sum_{\{t^1,t^2,\ldots,t^s\} \subseteq \Omega_{\tau^0}} \sum_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}} K_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}} \chi_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}(\sigma_{\Omega_{\tau^0}})\Big)$$

in this connection

$$p = \sum_{k=0}^{i_{\max}-1} (\sigma_{\tau^{i+k}} - 1) q^k,\ p = 0,1,2,\ldots, q^{i_{\max}} - 1$$

$$r=\sum_{k=0}^{i_{\max}-1}(\sigma_{\tau^{i+1+k}}-1)q^k,\quad r=0,1,2,\ldots,q^{i_{\max}}-1$$

Then

$$Z_L=\sum_{\{\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{L-1}}\}}\mathcal{T}_{\{(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}}),(\sigma_{\tau^1},\sigma_{\tau^2},\ldots,\sigma_{\tau^{i_{\max}}})\}}\cdot$$

$$\mathcal{T}_{\{(\sigma_{\tau^1},\sigma_{\tau^2},\ldots,\sigma_{\tau^{i_{\max}}}),(\sigma_{\tau^2},\sigma_{\tau^3},\ldots,\sigma_{\tau^{i_{\max}+1}})\}}\cdots$$

$$\mathcal{T}_{\{(\sigma_{\tau^{L-1}},\sigma_{\tau^0},\ldots,\sigma_{\tau^{i_{\max}-2}}),(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}})\}}=Tr(\mathcal{T}^L)$$

Note that nonzero matrix elements of the matrix $\mathcal{T}=\mathcal{T}_{p,r}$ will be equal if the set $(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}}})$ contains the same subset $\sigma(\Omega_{\tau^0})=\{\sigma_t,t\in\Omega_{\tau^0}\}$, with fixed spin values $\{\sigma_t,t\in\Omega_{\tau^0}\}$.

The eigenvector of the transfer matrix $\mathcal{T}=\mathcal{T}_{p,r}$, corresponding the higher eigenvalue F matrix seek in the formula $\vec{b}=\{b_{(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}})},\sigma_{\tau^i}\in X,i=1,\ldots,i_{\max}\}=\{b_p,p=\sum_{k=0}^{i_{\max}-1}(\sigma_{\tau^k}-1)q^k\}$, at the same time we consider that $b_{(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}})}=\hat{b}(\sigma_{\{\Omega_{\tau^0}\setminus\tau^{i_{\max}}\}})$,

that is, the components of the vector $\vec{b}$ will be the same, if the set $(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}})$ contains the same subset $\sigma(\tilde{\Omega}_{\tau^0})$ (the total number of various components $\hat{b}(\sigma(\tilde{\Omega}_{\tau^0}))$ will be $q^{|\tilde{\Omega}_{\tau^0}|}$). By the Perron-Frobenius theorem, for the maximum eigenvalue F all components of the vector $\vec{b}$ must be non-negative.

In this case, we can write the system of equations for the eigenvector of the matrix $\mathcal{T}=\mathcal{T}_{p,r}$, assuming that the components of the eigenvector $\{\hat{b}(\sigma_{\tilde{\Omega}_{\tau^0}})\}$ and the interaction coefficients $K_{\propto_{t^1},\propto_{t^2},\ldots,\propto_{t^s}}$ in the Hamiltonian are unknown:

$$b_{\{(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}})\}}F=\sum_{\sigma_{\tau^{i_{\max}}}\in X}\mathcal{T}_{\{(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}}),(\sigma_{\tau^1},\sigma_{\tau^2},\ldots,\sigma_{\tau^{i_{\max}}})\}}(\{K_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}\})b_{\{(\sigma_{\tau^1},\sigma_{\tau^2},\ldots,\sigma_{\tau^{i_{\max}}})\}}\tag{19}$$

where F - higher eigenvalue of an elementary transfer matrix $\mathcal{T}=\mathcal{T}_{p,r}$ (we assume that all components of the eigenvector $\vec{b}=\{b_{(\sigma_{\tau^1},\sigma_{\tau^2},\ldots,\sigma_{\tau^{i_{\max}}})},\sigma_{\tau^i}\in X,i=1,\ldots,i_{\max}\}$ are positive, then, by the Perron – Frobenius theorem, this vector will correspond to the highest eigenvalue). Rewrite (19) as

$$\hat{b}(\sigma_{\tilde{\Omega}_{\tau^0}})F=\sum_{\sigma_{\tau^{i_{\max}}}\in X}\mathcal{T}_{\{(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}}),(\sigma_{\tau^1},\sigma_{\tau^2},\ldots,\sigma_{\tau^{i_{\max}}})\}}\hat{b}(\sigma_{\tilde{\Omega}_{\tau^1}})$$

Where $\tilde{\Omega}_{\tau^1}=\tilde{\Omega}_{\tau^0}+\tau^1$.

Or $$\hat{b}(\sigma_{\tilde{\Omega}_{\tau^0}})F=\sum_{\sigma_{\tau^{i_{\max}}}\in X}\exp\Big(\sum_{\{t^1,t^2,\ldots,t^s\}\subseteq\Omega_{\tau^0}}\sum K_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}\chi_{\mu_{t^1},\mu_{t^2},\ldots,\mu_{t^s}}(\sigma_{\Omega_{\tau^0}})\Big)\hat{b}(\sigma_{\tilde{\Omega}_{\tau^1}})$$

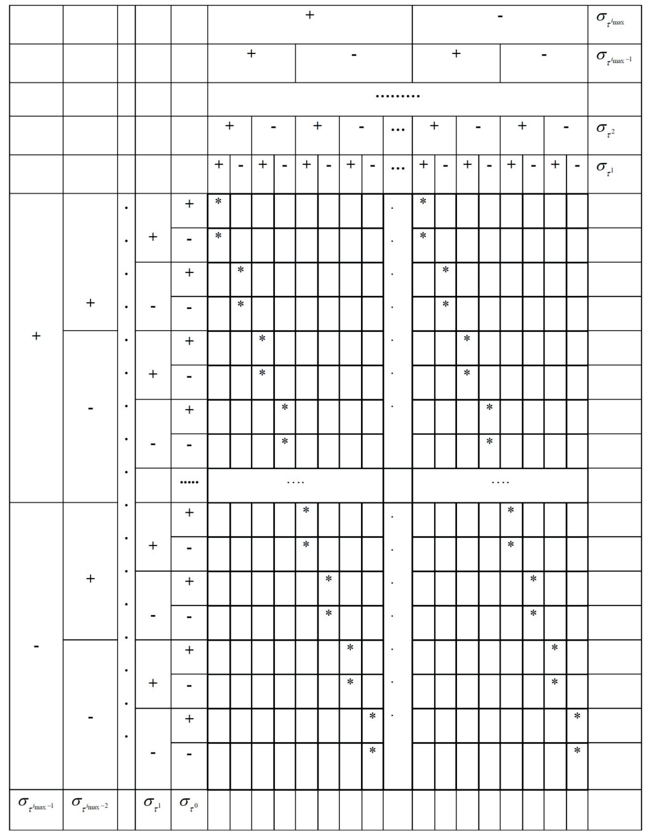

Fig. 1. The structure of the transfer matrix $\mathcal{T}_{\{(\sigma_{\tau^0},\sigma_{\tau^1},\ldots,\sigma_{\tau^{i_{\max}-1}}),(\sigma_{\tau^1},\sigma_{\tau^2},\ldots,\sigma_{\tau^{i_{\max}}})\}}$.
Asterisks (*) denote nonzero matrix elements. The sign "+" corresponds to the value of spin +1, the sign "-" the value of spin -1

Next, we analyze various examples of finding free energy.

2D Ising Model

Consider a two-dimensional square grid of size $L=L_1\times L_2$, total number of grid points $L=L_1L_2$, with special boundary cyclic screw (with a shift) conditions (1) and points renumbering (2). We assume that there is a particle in each node. The state of a particle is determined by its magnitude (spin) σ_i, which can take 2 values: +1 or -1. Each spin interacts with the eight nearest spins in four directions or lines. Hamiltonian model has the form

$$\mathcal{H}(\sigma)=-\sum_{m=1}^{L_2}\sum_{n=1}^{L_1}(J_1\sigma_n^m\sigma_{n+1}^m+J_2\sigma_n^m\sigma_n^{m+1}+J_3\sigma_{n+1}^m\sigma_n^{m+1}+J_4\sigma_n^m\sigma_{n+1}^{m+1}+J_5\sigma_n^m\sigma_n^{m+1}\sigma_{n+1}^{m+1}+J_6\sigma_n^m\sigma_{n+1}^m\sigma_n^{m+1}+J_7\sigma_n^m\sigma_{n+1}^m\sigma_{n+1}^{m+1}+J_8\sigma_n^{m+1}\sigma_{n+1}^m\sigma_{n+1}^{m+1}+J_9\sigma_n^m\sigma_{n+1}^m\sigma_n^{m+1}\sigma_{n+1}^{m+1}+h\sigma_n^m)\tag{20}$$

where J_i, $i=1,2,\ldots,9$ - corresponding coefficients of interspin interaction. We introduce $K_i=J_i/(k_BT)$, $i=1,2,\ldots,9$, where T - temperature, k_B - Boltzmann constant,

$H = h/(k_B T)$ interaction parameter with the external field with a coefficient. h. Then the statistical sum of the model can be written as

$$Z_{L_1 L_2} = \sum_{\sigma} \exp(-\mathscr{H}(\sigma)/k_B T) = \sum_{\sigma} \exp(\sum_{m=1}^{L_2} \sum_{n=1}^{L_1} (K_1 \sigma_n^m \sigma_{n+1}^m + K_2 \sigma_n^m \sigma_n^{m+1} + K_3 \sigma_{n+1}^m \sigma_n^{m+1} + K_4 \sigma_n^m \sigma_{n+1}^{m+1} + K_5 \sigma_n^m \sigma_n^{m+1} \sigma_{n+1}^{m+1} + K_6 \sigma_n^m \sigma_{n+1}^m \sigma_n^{m+1} + K_7 \sigma_n^m \sigma_{n+1}^m \sigma_{n+1}^{m+1} + K_8 \sigma_n^{m+1} \sigma_{n+1}^m \sigma_{n+1}^{m+1} + K_9 \sigma_n^m \sigma_{n+1}^m \sigma_n^{m+1} \sigma_{n+1}^{m+1} + H \sigma_n^m)) \quad (21)$$

where summation is performed over all states of spins.

For the model under consideration, we construct an elementary transfer matrix $\mathscr{T} = \mathscr{T}_{p,r}$ according to the formula (9) (Fig. 2). Let

$$G(\sigma_{\tau^0}, \sigma_{\tau^1}, \sigma_{\tau^{L_1-1}}, \sigma_{\tau^{L_1}}, K_1, K_2, \ldots, K_9, H) = \exp(K_1 \sigma_{\tau^0} \sigma_{\tau^1} + K_2 \sigma_{\tau^0} \sigma_{\tau^{L_1-1}} + K_3 \sigma_{\tau^1} \sigma_{\tau^{L_1-1}} + K_4 \sigma_{\tau^0} \sigma_{\tau^{L_1}} + K_5 \sigma_{\tau^0} \sigma_{\tau^{L_1-1}} \sigma_{\tau^{L_1}} + K_6 \sigma_{\tau^0} \sigma_{\tau^1} \sigma_{\tau^{L_1-1}} + K_7 \sigma_{\tau^0} \sigma_{\tau^1} \sigma_{\tau^{L_1}} + K_8 \sigma_{\tau^{L_1-1}} \sigma_{\tau^1} \sigma_{\tau^{L_1}} + K_9 \sigma_{\tau^0} \sigma_{\tau^1} \sigma_{\tau^{L_1-1}} \sigma_{\tau^{L_1}} + H \sigma_{\tau^0}) \quad (22)$$

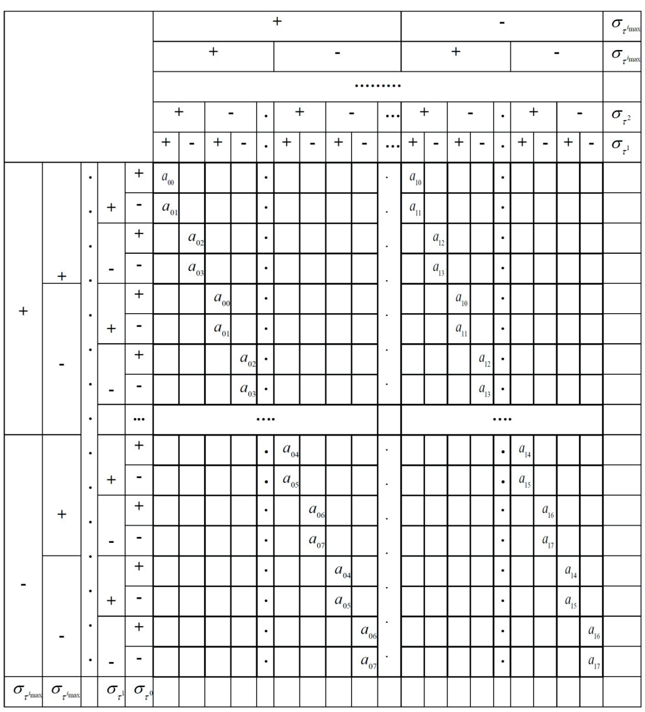

Fig. 2. Elementary transfer matrix for two-dimensional grid models with Hamiltonian (20).

Nonzero Matrix Elements a_{ij}, $i = 0,1,\ldots,7$, $j = 0,1$ of elementary transfer matrix $\mathscr{T} = \mathscr{T}_{p,r}$ we write in the form

$$a_{0i} = G(+1,+1,+1,1-2i, K_1, K_2, \ldots, K_9, H) \quad (23)$$
$$a_{1i} = G(-1,+1,+1,1-2i, K_1, K_2, \ldots, K_9, H)$$
$$a_{2i} = G(+1,-1,+1,1-2i, K_1, K_2, \ldots, K_9, H)$$
$$a_{3i} = G(-1,-1,+1,1-2i, K_1, K_2, \ldots, K_9, H)$$
$$a_{4i} = G(+1,+1,-1,1-2i, K_1, K_2, \ldots, K_9, H)$$
$$a_{5i} = G(-1,+1,-1,1-2i, K_1, K_2, \ldots, K_9, H)$$
$$a_{6i} = G(+1,-1,-1,1-2i, K_1, K_2, \ldots, K_9, H)$$
$$a_{7i} = G(-1,-1,-1,1-2i, K_1, K_2, \ldots, K_9, H)$$

a) Consider a special case when

$$J_5 = J_6 = J_7 = J_8 = H = 0 \cdot \quad (24)$$

Since in this case, replacing the signs of all spins with opposite values of the Hamiltonian (20) and the function $G(\sigma_{\tau^0}, \sigma_{\tau^1}, \sigma_{\tau^{L_1-1}}, \sigma_{\tau^{L_1}}, K_1, K_2, \ldots, K_9, H)$ does not change, so in this case the transfer matrix $\mathscr{T} = \mathscr{T}_{p,r}$ is centrally symmetric (Fig. 2). We will also search in a symmetric form for its own vector corresponding to the highest eigenvalue F.

$$\vec{x} = (\underbrace{1, b_2, 1, b_2, \ldots, 1, b_2}_{2^L}, \underbrace{b_2, 1, b_2, 1, \ldots, b_2, 1}_{2^L})^T \quad (25)$$

where $b_2 > 0$. Then, by the Perron-Frobenius theorem, this eigenvector will correspond to the maximum eigenvalue of F. Denote

$$R_i = \exp(K_i), \quad i = 1,2,\ldots,9. \quad (26)$$

From the form of the transfer matrix (Fig. 2), we obtain the following system of equations for R_i, $i = 1,2,3,4,9$, F, b_2:

$$F = R_1 R_2 R_3 (R_4 R_9 + b_2/(R_4 R_9)) \quad \text{(a)}$$
$$b_2 F = (R_3/(R_1 R_2))(1/(R_4 R_9) + b_2 R_4 R_9) \quad \text{(b)} \quad (27)$$
$$F = (R_2/(R_1 R_3))(b_2 R_4/R_9 + R_9/R_4) \quad \text{(c)}$$
$$b_2 F = (R_1/(R_2 R_3))(b_2 R_9/R_4 + R_4/R_9) \quad \text{(d)}$$

Solve the system (27).
We equate the right sides (27a) and (27c),

$$R_1 R_2 R_3 (R_4 R_9 + b_2/(R_4 R_9)) = (R_2/(R_1 R_3))(b_2 R_4/R_9 + R_9/R_4) \quad (28)$$

find from here b_2:

$$b_2 = -(-1 + R_1^2 R_3^2 R_4^2) R_9^2 / (R_1^2 R_3^2 - R_4^2) \quad (29)$$

We equate the right sides (27b) and (27d),

$$R_3/(R_1 R_2)(1/(R_4 R_9) + b_2 R_4 R_9) = (R_1/(R_2 R_3))(b_2 R_9/R_4 + R_4/R_9) \quad (30)$$

Then b_2:

$$b_2 = (R_3^2 - R_1^2 R_4^2)/((R_1^2 - R_3^2 R_4^2) R_9^2) \quad (31)$$

We equate the b_2 from (29) and (31):

$$-(-1 + R_1^2 R_3^2 R_4^2) R_9^2 / (R_1^2 R_3^2 - R_4^2) = (R_3^2 - R_1^2 R_4^2)/((R_1^2 - R_3^2 R_4^2) R_9^2) \quad (32)$$

Solve for R_9,

Result

$$R_9^4=\frac{(R_3^2-R_1^2R_4^2)(R_1^2R_3^2-R_4^2)}{(R_1^2-R_3^2R_4^2)(1-R_1^2R_3^2R_4^2)} \quad (33)$$

Substitute R_9^2 from (33) (2.10) to b_2 from (29) (2.6), result

$$b_2=\frac{(1-R_1^2R_3^2R_4^2)}{(R_1^2R_3^2-R_4^2)}\sqrt{\frac{(R_3^2-R_1^2R_4^2)(R_1^2R_3^2-R_4^2)}{(R_1^2-R_3^2R_4^2)(1-R_1^2R_3^2R_4^2)}} \quad (34)$$

Multiplying the equation (27a)(2.4a) times b_2, the right-hand side is equated to the right-hand side of the equation from (27b) (2.4b)

$$b_2R_1R_2R_3(R_4R_9+b_2/(R_4R_9))= \quad (35)$$
$$(R_3/(R_1R_2))(1/(R_4R_9)+b_2R_4R_9)$$

Solve for R_2, result

$$R_2^2=\frac{(R_3/R_1)(1/(R_4R_9)+b_2R_4R_9)}{b_2R_1R_3(R_4R_9+b_2/(R_4R_9))} \quad (36)$$

Substitute formula for R_9 from (33) and for b_2 from (34) into (36), result

$$R_2^2=\frac{R_1^2R_3^2-R_4^2}{R_3^2-R_1^2R_4^2} \quad (37)$$

Result

$$b_2=\frac{(1-R_1^2R_3^2R_4^2)}{(R_1^2R_3^2-R_4^2)}\sqrt{\frac{(R_3^2-R_1^2R_4^2)(R_1^2R_3^2-R_4^2)}{(R_1^2-R_3^2R_4^2)(1-R_1^2R_3^2R_4^2)}} \quad (38)$$

$$R_2=\sqrt{\frac{R_1^2R_3^2-R_4^2}{R_3^2-R_1^2R_4^2}} \quad (39)$$

$$R_9=\left(\frac{(R_3^2-R_1^2R_4^2)(R_1^2R_3^2-R_4^2)}{(R_1^2-R_3^2R_4^2)(1-R_1^2R_3^2R_4^2)}\right)^{\frac{1}{4}} \quad (40)$$

or

$$b_2=\frac{-\sinh(K_1+K_3+K_4)}{\sinh(K_1+K_3-K_4)}\sqrt{\frac{\sinh(K_1-K_3+K_4)\sinh(K_1+K_3-K_4)}{\sinh(K_1-K_3-K_4)\sinh(K_1+K_3+K_4)}} \quad (41)$$

$$R_2=\sqrt{\frac{\sinh(K_1+K_3-K_4)}{\sinh(-K_1+K_3-K_4)}} \quad (42)$$

$$R_9=\left(\frac{\sinh(-K_1+K_3-K_4)\sinh(K_1+K_3-K_4)}{\sinh(K_1-K_3-K_4)\sinh(-K_1-K_3-K_4)}\right)^{\frac{1}{4}} \quad (43)$$

On the region of existence of solutions we get the system of inequation:

$$(K_1+K_3+K_4)(K_1+K_3-K_4)(K_1-K_3-K_4)(K_1+K_3+K_4)>0$$

$$(K_1+K_3+K_4)(K_1+K_3-K_4)<0 \quad (44)$$

$$(K_1+K_3-K_4)(-K_1+K_3-K_4)>0$$

$$(-K_1+K_3-K_4)(K_1+K_3-K_4)(K_1-K_3-K_4)(-K_1-K_3-K_4)>0$$

It is tantamount to:

$$\begin{cases}(|K_1+K_3|-|K_4|)(|K_1|-|K_3+K_4|)>0\\ |K_1+K_3|-|K_4|<0\\ |K_3-K_4|-|K_1|>0\\ (|K_3-K_4|-|K_1|)(|K_3+K_4|-|K_1|)>0\end{cases} \quad (45)$$

The last inequation is a consequence of the first three. Therefore

$$\begin{cases}(|K_1|-|K_3+K_4|)<0\\ |K_1+K_3|-|K_4|<0\\ |K_3-K_4|-|K_1|>0\end{cases} \quad (46)$$

Or

$$\begin{cases}|K_1|<|K_3+K_4|\\ |K_1+K_3|<|K_4|\\ |K_3-K_4|>|K_1|\end{cases} \quad (47)$$

It is tantamount to:

$$|K_1+K_3|<|K_4| \quad (48)$$

In this domain of formulas definition, you can still simplify the formula for b2:

$$b_2=\sqrt{\frac{\sinh(K_1+K_3+K_4)\sinh(K_1-K_3+K_4)}{\sinh(-K_1-K_3+K_4)\sinh(K_1-K_3-K_4)}} \quad (49)$$

Example.

$K_1=-1.5$, $K_3=-0.65$, $K_4=2.2$,

$R_1=0.22313016014842982$, $R_3=0.522045776761016$,

$R_2=4.639942095766785$, $R_4=9.025013499434122$,

$K_2=1.5347018867996893$, $R_9=3.391150413013755$,

$K_9=1.2211692186958323$, $b_2=0.01485136611090205$,

$F=16.541741276496538=\exp(2.8058869607477605612)$.

It follows that, the free energy of the system

$f=2.8058869607477605612$.

It is constant, and does not depend on $L_1>2$ and $L_2>2$.

Consider a two-dimensional square lattice of size $L=L_1\times L_2$, with the Hamiltonian (20) this time in general form, without restrictions of the form (24)

The eigenvector of the elementary transfer matrix $\mathcal{T}=\mathcal{T}_{p,r}$ (Fig. 2) we find in the shape of

$$\vec{x}=(\underbrace{1,b_1,1,b_1,\dots,1,b_1}_{2^{L_1}};\underbrace{b_2,b_3,b_2,b_3,\dots,b_2,b_3}_{2^{L_1}}),b_i\ge 0,i=1,2,3. \quad (50)$$

Then in order to make $\vec{x}$ an eigenvector, we get the system of equations

$$\begin{cases} F = a_{00} + b_2 a_{01} \\ b_1 F = a_{10} + b_2 a_{11} \\ F = b_1 a_{20} + b_3 a_{21} \\ b_1 F = b_1 a_{30} + b_3 a_{31} \\ b_2 F = a_{40} + b_2 a_{41} \\ b_3 F = a_{50} + b_2 a_{51} \\ b_2 F = b_1 a_{60} + b_3 a_{61} \\ b_3 F = b_1 a_{70} + b_3 a_{71} \end{cases} \quad (51)$$

where F - eigenvalue of the transfer matrix $\mathcal{T} = \mathcal{T}_{p,r}$. $b_i > 0, i = 1,2,3$, then F is the highest (maximum) eigenvalue of the transfer matrix τ (and $f = \ln(F)$ - free energy).
The system of equations (51) is rewritten as

$$\begin{cases} F = \exp(H + K1 + K2 + K3 + K4 + K5 + K6 + K7 + K8 + K9) + \\ b_2 \exp(H + K1 + K2 + K3 - K4 - K5 + K6 - K7 - K8 - K9) \\ b_1 F = \exp(-H - K1 - K2 + K3 - K4 - K5 - K6 - K7 + K8 - K9) + \\ b_2 \exp(-H - K1 - K2 + K3 + K4 + K5 - K6 + K7 - K8 + K9) \\ F = b_1 \exp(H - K1 + K2 - K3 + K4 + K5 - K6 - K7 - K8 - K9) + \\ b_3 \exp(H - K1 + K2 - K3 - K4 - K5 - K6 + K7 + K8 + K9) \\ b_1 F = b_1 \exp(-H + K1 - K2 - K3 - K4 - K5 + K6 + K7 - K8 + K9) + \\ b_3 \exp(-H + K1 - K2 - K3 + K4 + K5 + K6 - K7 + K8 - K9) \\ b_2 F = \exp(H + K1 - K2 - K3 + K4 - K5 - K6 + K7 - K8 - K9) + \\ b_2 \exp(H + K1 - K2 - K3 - K4 + K5 - K6 - K7 + K8 + K9) \\ b_3 F = \exp(-H - K1 + K2 - K3 - K4 + K5 + K6 - K7 - K8 + K9) + \\ b_2 \exp(-H - K1 + K2 - K3 + K4 - K5 + K6 + K7 + K8 - K9) \\ b_2 F = b_1 \exp(H - K1 - K2 + K3 + K4 - K5 + K6 - K7 + K8 + K9) + \\ b_3 \exp(H - K1 - K2 + K3 - K4 + K5 + K6 + K7 - K8 - K9) \\ b_3 F = b_1 \exp(-H + K1 + K2 + K3 - K4 + K5 - K6 + K7 + K8 - K9) + \\ b_3 \exp(-H + K1 + K2 + K3 + K4 - K5 - K6 - K7 - K8 + K9) \end{cases} \quad (52)$$

System (52) of 8 equations has 14 variables. So, the solution of system (52) in the general case depends on 6 parameters. In order to show that system (52) has nontrivial solutions, we indicate the solution found by the Levenberg-Marquardt method:

K_1 =-0.072508931144562,

K_2 =0.227087903394048,

K_3 =-0.033524527164916,

K_4 =0.309028362673801,

K_5 =0.081977911068159,

K_6 =-0.085603418100829,

K_7 =-0.135244289148472,

K_8 =0.228449686392315,

K_9 =0.112019459850970,

H =0.010568103316912,

$b_1 = 0.611722309307112$,

$b_2 = 0.572612052286677$,

$b_3 = 0.861858834747567$

Maximum eigenvalue by Perron-Frobenius theorem
F =2.231051645841576, free energy f = 0.802473064371868.

c) Let us analyze a two-dimensional square grid of size $L = L_1 \times L_2$, with Hamiltonian (20) in general terms. The eigenvector of the elementary transfer matrix $\mathcal{T} = \mathcal{T}_{p,r}$ (Fig. 2) we will find in the form

$$\vec{x} = (\underbrace{1, b_1, b_2, b_3, \ldots, 1, b_1, b_2, b_3}_{2^{L_1}}; \underbrace{b_4, b_5, b_6, b_7, \ldots, b_4, b_5, b_6, b_7}_{2^{L_1}}), b_i \geq 0, i = 1,2,\ldots,7. \quad (53)$$

Then in order $\vec{x}$ to be an eigenvector, we get a system of 16 equations

$$\begin{cases} F = a_{00} + b_4 a_{01} \\ b_1 F = a_{10} + b_4 a_{11} \\ b_2 F = b_1 a_{20} + b_5 a_{21} \\ b_3 F = b_1 a_{30} + b_5 a_{31} \\ F = b_2 a_{00} + b_6 a_{01} \\ b_1 F = b_2 a_{10} + b_6 a_{11} \\ b_2 F = b_3 a_{20} + b_7 a_{21} \\ b_3 F = b_3 a_{30} + b_7 a_{31} \\ b_4 F = a_{40} + b_4 a_{41} \\ b_5 F = a_{50} + b_4 a_{51} \\ b_6 F = b_1 a_{60} + b_5 a_{61} \\ b_7 F = b_1 a_{70} + b_5 a_{71} \\ b_4 F = b_2 a_{40} + b_6 a_{41} \\ b_5 F = b_2 a_{50} + b_6 a_{51} \\ b_6 F = b_3 a_{60} + b_7 a_{61} \\ b_7 F = b_3 a_{70} + b_7 a_{71} \end{cases} \quad (54)$$

where F - eigenvalue of the transfer matrix $\mathcal{T} = \mathcal{T}_{p,r}$. By the Perron-Frobenius theorem in this case (if $b_i > 0, i = 1,2,3.$) F is the maximum eigenvalue of the transfer matrix $\mathcal{T} = \mathcal{T}_{p,r}$ (then $f = \ln(F)$ - free energy). This system is formally wider than the general system of equations from the beginning of the article (in this case it was necessary to restrict ourselves to a system of 8 equations), but it has nontrivial solutions (the solution was found by the Levenberg-Marquardt method), and this example shows the possibilities of expanding the search area for exact solutions:

K_1 =0.019113809839761

K_2 = -0.233807683866557

K_3 = -0.034832967333049

K_4 = -0.232974813199571

K_5 =0.339920870267006

K_6 = 0.117189864650398

K_7 =0.232974813199571

K_8 = -0.000000000000000

K_9 = -0.339920870267006

H = 0.150660735316256

b_1 = 0.899142765909877

b_2 =0.600000000000000 (it is specifically a priori taken so not to coincide with the solution of point b))

b_3 = 0.899142765909877

b_4 = 1.353781209394591

b_5 = 0.763468696132035

b_6 =1.753781209394591

b_7 = 0.763468696132035

F =2.3973089056900103166, f = 0.8743468189429043358.

Note that $K_4 = -K_7$, $K_5 = -K_9$, $K_8 = 0$, $K_{12} = b_2 = 0.6$ (So b2 was given to get a more general form of the eigenvector than in the three-parameter case b1, b2, b3), $K_{13} = K_{11}$ ($K_{13} = b_3, K_{11} = b_1$), $K_{15} = K_{17}$. I.e $b_1 = b_3$, $b_5 = b_7$.
The following example is similar to the previous one:

K_1 = 0.174718289126992

K_2 =-0.796130400068089

K_3 =0.036359406404343

K_4 =0.187050114181306

K_5 = -0.191302175159284

K_6 =-0.050020923787075

K_7 =-0.187050114188515

K_8 =0.000000000036532

K_9 =0.191302175154890

H =-0.058314486090843

b_1 =4.303785746876578

b_2 =0.600000000006545

b_3 =4.303785746875397

b_4 =5.050987834488805

b_5 =0.736735440877358

b_6 =5.450987834486589

b_7 =0.736735440878770

$b_5 = b_7$, $b_1 = b_3$, F = 3.024765061088569,

f =1.106833422710819 free energy.

ν - dimensional Ising model

Let us concretize the system of equations (17), generated by the model with Hamiltonian (3) for the ν -dimensional case, the interaction within the framework of a single -dimensional cube $\Omega = \{t = (t_1, t_2, \ldots, t_\nu) : t_i = 0,1, i = 1,2,\ldots,\nu\}$

$$\hat{b}(\sigma_{\tilde{\Omega}_{\tau 0}})F = \sum_{\sigma_{\tau^{i_{\max}}} \in X} \exp\Big(\sum_{\{t^1,t^2,\ldots,t^s\} \subseteq \Omega_{\tau 0}} K_{t^1,t^2,\ldots,t^s} \sigma_{t^1}\sigma_{t^2}\ldots\sigma_{t^s}\Big)\hat{b}(\sigma_{\tilde{\Omega}_{\tau 1}}) \quad (55)$$

Then

$$\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2+1}},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}}},\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+1}},\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1}}\}} F =$$

$$\sum \mathscr{T}_{\{(\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1}}),(\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2+1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1+1}})\}} \cdot$$

$$\hat{b}_{\{\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2+1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1+1}}\}}$$

where is the summation over $\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1+1}} = \pm 1$.

Or

$$\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2+1}},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}}},\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+1}},\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1}}\}} F = \quad (56)$$

$$\sum_{\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1+1}}=\pm 1} \exp\Big(\sum_{\{t^1,t^2,\ldots,t^s\}\subseteq\Omega_{\tau_0}} K_{t^1,t^2,\ldots,t^s}\sigma_{t^1}\sigma_{t^2}\ldots\sigma_{t^s}\Big).$$

$$\hat{b}_{\{\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2+1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1+1}}\}}$$

where is the summation over all non-empty subsets $\{t^1, t^2, \ldots, t^s\} \subseteq \Omega_{\tau_0}$. The number of such subsets for ν - dimensional cube will be 2^{2^ν} , total different components $\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2+1}},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}}},\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+1}},\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1}},\ldots,\sigma_{\tau_{L_1L_2\ldots L_{\nu-1}+L_1L_2\ldots L_{\nu-2}+\ldots+L_1}}\}}$ -($2^{2^\nu-1} - 1$) (we assume the very first component equal to 1 for normalizing the eigenvector). In all system (56) has $2^{2^\nu} + 2^{2^\nu-1} - 1$

equations (256+127=383 in the three-dimensional case). The system of equations (56) is perfectly solved by the Levenberg-Marquardt method. In order to unify the system of equations for solving by the Levenberg-Marquardt method, to obtain only positive coefficients

$$\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}},\dots,\sigma_{\tau_{L_1L_2\dots L_{\nu-1}}},\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+1}},\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1}},\dots,\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1L_2\dots L_{\nu-2}+\dots+L_1}}\}}$$

and not to introduce restrictions on the field of using the Levenberg-Marquardt method, we introduce

$K_{t^1,t^2,\dots,t^s} = K_{i(t^1,t^2,\dots,t^s)}$, where $i(t^1,t^2,\dots,t^s)$ - decimal notation of binary 2^ν – digit number, that on the t^p -th position, $p=1,\dots,s$, $s=1,\dots,2^\nu$ has 1, and the rest are zeros. $i(t^1,t^2,\dots,t^s)=1,\dots,2^{2^\nu}$.

The components of the eigenvector with the highest eigenvalue are expressed as

$$\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}},\dots,\sigma_{\tau_{L_1L_2\dots L_{\nu-1}}},\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+1}},\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1}},\dots,\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1L_2\dots L_{\nu-2}+\dots+L_1}}\}} =$$

$$\exp(K_{i(\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}},\dots,\sigma_{\tau_{L_1L_2\dots L_{\nu-1}}},\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+1}},\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1}},\dots,\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1L_2\dots L_{\nu-2}+\dots+L_1}})})$$

Where

$$i(\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}},\dots,\sigma_{\tau_{L_1L_2\dots L_{\nu-1}}},\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+1}},\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1}},\dots,\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1L_2\dots L_{\nu-2}+\dots+L_1}}) =$$
$$2^{2^\nu} + \frac{(1-\sigma_{\tau_0})}{2}2^0 + \frac{(1-\sigma_{\tau_1})}{2}2^1 + \frac{(1-\sigma_{\tau_{L_1}})}{2}2^2 + \frac{(1-\sigma_{\tau_{L_1+1}})}{2}2^3 + \frac{(1-\sigma_{\tau_{L_1L_2}})}{2}2^4 + \frac{(1-\sigma_{\tau_{L_1L_2+1}})}{2}2^5 + \frac{(1-\sigma_{\tau_{L_1L_2+1}})}{2}2^6 + \dots + \frac{(1-\sigma_{\tau_{L_1L_2\dots L_{\nu-1}+L_1L_2\dots L_{\nu-2}+\dots+L_1}})}{2}2^{2^\nu-1}$$

When $\nu=3$ we have:

$$\Omega = \{t=(t_1,t_2,t_3): t_i = 0,1, i=1,2,3\}$$

The system (55) is without changes.
Then

$$\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}}\}}F = \sum_{\sigma_{\tau_{L_1L_2+L_1+1}}=\pm1} \mathscr{T}_{\{(\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}}),(\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}})\}} \cdot \hat{b}_{\{\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}}\}} \tag{57}$$

Or

$$\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}}\}}F = \sum_{\sigma_{\tau_{L_1L_2+L_1+1}}=\pm1} \exp\Big(\sum_{\{t^1,t^2,\dots,t^s\}\subseteq\Omega_{\tau_0}} K_{t^1,t^2,\dots,t^s}\sigma_{t^1}\sigma_{t^2}\dots\sigma_{t^s}\Big) \hat{b}_{\{\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}},\sigma_{\tau_{L_1L_2+L_1+1}}\}} \tag{58}$$

where summation over all non-empty subsets $\{t^1,t^2,\dots,t^s\}\subseteq\Omega_{\tau_0}$. The number of such subsets for a 3-dimensional cube will be $2^{2^3} = 2^8 = 256$, the total number of different components $\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}}\}}$ will be $2^{2^3-1}-1$ =127 (we consider the very first component equal to 1 for normalizing the eigenvector). Total system (58) has 256 + 127 = 383 equations. System of equations (58) is perfectly solved by the Levenberg-Marquardt method. In order to unify the system of equations for solving by the Levenberg-Marquardt method, to obtain only positive coefficients $\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}}\}}$ and not to introduce restrictions on the field of using the Levenberg-Marquardt method, we introduce

$K_{t^1,t^2,\dots,t^s} = K_{i(t^1,t^2,\dots,t^s)}$, where $i(t^1,t^2,\dots,t^s)$ - decimal notation of binary 2^3 =8– digit number, that on the t^p -th position, $p=1,\dots,s$, $s=1,\dots,2^3$ has 1, and the rest are zeros. $i(t^1,t^2,\dots,t^s)=1,\dots,2^{2^3}$. The components of the eigenvector with the highest eigenvalue are expressed as

$$\hat{b}_{\{\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}}\}} = \exp(K_{i(\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2}+1},\sigma_{\tau_{L_1L_2+L_1}})}) \tag{59}$$

where

$$i(\sigma_{\tau_0},\sigma_{\tau_1},\sigma_{\tau_{L_1}},\sigma_{\tau_{L_1+1}},\sigma_{\tau_{L_1L_2}},\sigma_{\tau_{L_1L_2+1}},\sigma_{\tau_{L_1L_2+L_1}}) = 2^{2^3} + \frac{(1-\sigma_{\tau_0})}{2}2^0 + \frac{(1-\sigma_{\tau_1})}{2}2^1 + \frac{(1-\sigma_{\tau_{L_1}})}{2}2^2 + \frac{(1-\sigma_{\tau_{L_1+1}})}{2}2^3 + \frac{(1-\sigma_{\tau_{L_1L_2}})}{2}2^4 + \frac{(1-\sigma_{\tau_{L_1L_2+1}})}{2}2^5 + \frac{(1-\sigma_{\tau_{L_1L_2+L_1}})}{2}2^6 \tag{60}$$

At the initial approximation $K_i = 0.1$, $i=1,2,\dots,383$, by the Levenberg-Marquardt method, the iterative process converges to the exact solution. The author does not write out the solution itself, since the set of all coefficients recorded with high accuracy takes several sheets of text.

Let $\Omega = \{\tau_0,\tau_1,\tau_{L_1},\tau_{L_1L_2}\} = \{t_0,t_1,t_2,t_3\}$.

Hamiltonian of 3D model is equal to

$$\mathscr{H}(\sigma) = -\sum_{i=1}^{L_2}\sum_{n=1}^{L_1}(J_{01}\sigma_0\sigma_1 + J_{02}\sigma_0\sigma_2 + J_{03}\sigma_0\sigma_3 + J_{12}\sigma_1\sigma_2 + J_{13}\sigma_1\sigma_3 + J_{23}\sigma_2\sigma_3 + J_{0123}\sigma_0\sigma_1\sigma_2\sigma_3 \tag{61}$$

There is a one-to-one correspondence between the system of equations generated by Hamiltonian (61) and the system of equations for the flat model generated by Hamiltonian (20) with conditions (24), we search for the eigenvector in the form (25), this is clear from the correspondence of the vertices: $(n,m) \leftrightarrow \tau_0$, $(n+1,m) \leftrightarrow \tau_1$, $(n,m+1) \leftrightarrow \tau_{L_1}$, $(n+1,m+1) \leftrightarrow \tau_{L_1L_2}$.

At the same time, after reduction to a flat model, the statistical sum and free energy will depend only on the sum $J_{01} + J_{23}$ и $J_{02} + J_{13}$ (and not on each variable separately).

That is, for a three-dimensional model with Hamiltonian (61) there will be the same system of equations for the interaction coefficients and free energy, as for the flat model in paragraphs a) and b) with the interaction coefficients

$$J_1 = J_{01} + J_{23},\quad J_2 = J_{02} + J_{13},$$

$$J_3 = J_{12},\quad J_4 = J_{03},\quad J_5 = J_6 = J_7 = J_8 = 0,$$

$$J_9 = J_{0123},\quad h = 0\quad .$$

Summary

The calculation of the statistical sum of the Ising and Potts lattice models for various values of the parameters is still of scientific interest. Therefore, finding a set of disorder exact solutions is an important task. The article obtained a system of nonlinear equations for finding the exact value of the free energy of a wide class of models with finite spin space, and obtained solutions of this system, depending on several parameters, for a wide class of models. The existence domain of these solutions is obtained as well.

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