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Fourier Transform of Transfer Matrices of Plane Ising Models

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Abstract

This work demonstrates the Fourier transform of the elementary transfer matrix of the generalized two-dimensional Ising model with special boundary conditions with a shift (screw type) with the form of a Hamiltonian covering the classical Ising model with an external field, as well as models equivalent to models on a triangular lattice with a chessboard type Hamiltonian (the author plans to consider the general form of interaction in the following publication). Its limit representation is obtained in the form of a sum of integral operators with the size of the system tending to infinity. This allows the actual problem of finding the maximum eigenvalue of the limiting elementary transfer matrix (its Napierian logarithm is equal to the free energy of the system) to be brought to finding the maximum eigenvalue of the sum of integral operators of a fairly simple form. This approach can help solve the problems associated with the large size of the transfer matrices.

Keywords: generalized Ising model, Hamiltonian, elementary transfer matrix, partition function, free energy.

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Преобразование Фурье трансфер-матриц плоских моделей Изинга

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Аннотация
В работе сделано преобразование Фурье элементарной трансфер-матрицы обобщенной двумерной модели Изинга со специальными граничными условиями со сдвигом (типа винтовых) с видом гамильтониана, охватывающем классическую модель Изинга с внешним полем, а также модели, эквивалентные моделям на треугольной решетке с гамильтонианом «шахматного» типа (общий вид взаимодействия автор планирует рассмотреть в следующей публикации). Получено ее предельное представление в виде суммы интегральных операторов при размерах системы, стремящихся к бесконечности. Это позволяет актуальную задачу нахождения максимального собственного значения предельной элементарной трансфер-матрицы (ее натуральный логарифм равен свободной энергии системы) свести к нахождению максимального собственного значения суммы интегральных операторов достаточно простого вида. Такой подход может помочь снять проблемы, связанные с большим размером трансфер-матриц.

Ключевые слова: обобщенная модель Изинга, гамильтониан, элементарная трансфер-матрица, статистическая сумма, свободная энергия.

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Introduction

The transfer matrix method is successfully used to find partition functions, free energy, and other characteristics of physical systems [1-3]. The Fourier transform is often used when studying statistical physics models, for example, in [4], when finding spontaneous magnetization, both the transfer matrix and the Fourier transform are used. In [5], the Fourier transform of the correlation function of the Ising model in two and three dimensions is modelled. This work demonstrates the Fourier transform of the special elementary transfer matrix of the Ising model [6] with a Hamiltonian covering the classical Ising model with an external field, as well as models equivalent to models on a triangular lattice with a chessboard type Hamiltonian [7]. Its limit representation is obtained in the form of a sum of integral operators with the size of the system tending to infinity. This allows the actual problem of finding the maximum eigenvalue of the elementary transfer matrix [6] (its Napierian logarithm is equal to the free energy of the system) to be brought to finding the maximum eigenvalue of the sum of integral operators of a fairly simple form. It is possible that the eigenfunctions of this operator may also be generalized functions (such solutions are demonstrated in [6], they are for the more general form of the Hamiltonian, after the Fourier transform they will become generalized functions). By iterating approximations to the eigenfunction, for example, by the power iteration, we look for the eigenfunction of the transfer matrix immediately in an infinite volume, which is very significant and it can solve the problems associated with the finiteness of the volume under consideration in the calculations.

The Basic Definitions

Let us consider a two-dimensional square lattice of $L \times M$ size, the total number of nodes of the lattice is $R = LM$. So we will assume that there is a particle in each node. The state of a particle is determined by a σ_i that can take 2 values: +1 or -1. Each spin interacts with the eight nearest spins in four directions or lines. The Hamiltonian of the model has the form

$$\mathrm{H}(\sigma) = -\sum_{n=1}^{L}\sum_{m=1}^{M}(J_1\sigma_n^m\sigma_{n+1}^m + J_2\sigma_n^m\sigma_n^{m+1} + \\ = J_3\sigma_{n+1}^m\sigma_n^{m+1} + J_4\sigma_n^m\sigma_{n+1}^{m+1} + J_5\sigma_n^m\sigma_n^{m+1}\sigma_{n+1}^{m+1} + \\ +J_6\sigma_n^m\sigma_{n+1}^m\sigma_n^{m+1} + J_7\sigma_n^m\sigma_{n+1}^m\sigma_{n+1}^{m+1} + J_8\sigma_n^{m+1}\sigma_{n+1}^m\sigma_{n+1}^{m+1} + \\ +J_9\sigma_n^m\sigma_{n+1}^m\sigma_n^{m+1}\sigma_{n+1}^{m+1} + h\sigma_n^m), \quad (1)$$

where J_i, $i = 1,2,...,9$ - corresponding spin-spin interaction coefficients. We introduce $K_i = J_i / (k_B T)$, $i = 1,2,...,9$, where T - temperature, k_B - the Boltzmann constant, $H = h/(k_B T)$ interaction parameter with an external field with a coefficient h. Then the partition function of the model can be written in the following form

$$Z_{LM} = \sum_\sigma \exp(-\mathrm{H}(\sigma)/k_B T) = \sum_\sigma \exp(\sum_{n=1}^{L}\sum_{m=1}^{M}(K_1\sigma_n^m\sigma_{n+1}^m + \\ +K_2\sigma_n^m\sigma_n^{m+1} + K_3\sigma_{n+1}^m\sigma_n^{m+1} + K_4\sigma_n^m\sigma_{n+1}^{m+1} + K_5\sigma_n^m\sigma_n^{m+1}\sigma_{n+1}^{m+1} + \\ +K_6\sigma_n^m\sigma_{n+1}^m\sigma_n^{m+1} + K_7\sigma_n^m\sigma_{n+1}^m\sigma_{n+1}^{m+1} + K_8\sigma_n^{m+1}\sigma_{n+1}^m\sigma_{n+1}^{m+1} + \\ +K_9\sigma_n^m\sigma_{n+1}^m\sigma_n^{m+1}\sigma_{n+1}^{m+1} + H\sigma_n^m)), \quad (2)$$

where the summation is performed over all spin states.
Define the toroidal "winding" boundary conditions:

$$\sigma_n^{M+1} = \sigma_n^1, \quad n = 1,2,...,L,$$
$$\sigma_{L+1}^m = \sigma_1^{m+1}, \quad m = 1,2,...,M. \quad (3)$$

We renumber all the spins of the helical torus from $i = 1$ to $i = LM$.
Let us introduce the function

$$t(\sigma_i,\sigma_{i+1},...,\sigma_{i+L},\sigma_{i+L+1}) = \exp(K_1\sigma_i\sigma_{i+1} + K_2\sigma_i\sigma_{i+L} + \\ K_3\sigma_{i+1}\sigma_{i+L} + K_4\sigma_i\sigma_{i+L+1} + K_5\sigma_i\sigma_{i+L}\sigma_{i+L+1} + K_6\sigma_i\sigma_{i+1}\sigma_{i+L} + \\ K_7\sigma_i\sigma_{i+1}\sigma_{i+L+1} + K_8\sigma_{i+L}\sigma_{i+1}\sigma_{i+L+1} + K_9\sigma_i\sigma_{i+1}\sigma_{i+L}\sigma_{i+L+1} + H\sigma_i) \quad (4)$$

Then

$$Z_{LM} = \sum_\sigma \prod_{i=1}^{LM} t(\sigma_i,\sigma_{i+1},...,\sigma_{i+L},\sigma_{i+L+1}) \quad (5)$$

Let us introduce the elementary transfer matrix $A = A_{p,q}$ with the size $2^{L+1} \times 2^{L+1}$, nonzero elements $A_{p,q}$ re numbered with pairs of sets $\{(\sigma_i,\sigma_{i+1},...,\sigma_{i+L}),(\sigma_{i+1},...,\sigma_{i+L},\sigma_{i+L+1})\}$, in this connection

$$p = \sum_{k=0}^{L}\frac{(1-\sigma_{i+k})}{2}2^k, \quad p = 0,1,2,...,2^{L+1}-1, \quad (6)$$

$$q = \sum_{k=0}^{L}\frac{(1-\sigma_{i+1+k})}{2}2^k, \quad q = 0,1,2,...,2^{L+1}-1. \quad (7)$$

Then

$$Z_{LM} = Tr(A^{LM}). \quad (8)$$

Formula (8) shows the conformity of the term "elementary transfer matrix" with the classical definition of the transfer matrix and its use: the task of finding the partition function and the free energy of the system is reduced to finding the maximum eigenvalue $\lambda_{\max}$ of the elementary transfer matrix $A = A_{p,q}$, the free energy is $f = \ln(\lambda_{\max})$. Figure 1 shows the form of an elementary transfer matrix at $L = 2$.

L=2			+				-				σ_{i+3}
			+		-		+		-		σ_{i+2}
			+	-	+	-	+	-	+	-	σ_{i+1}
+	+	+	a_{00}				a_{01}				
		-	a_{10}				a_{11}				
	-	+		a_{20}				a_{21}			
		-		a_{30}				a_{31}			
-	+	+			a_{40}				a_{41}		
		-			a_{50}				a_{51}		
	-	+				a_{60}				a_{61}	
		-				a_{70}				a_{71}	
σ_{i+2}	σ_{i+1}	σ_i									

F i g. 1. Type of elementary transfer matrix $A = A_{p,q}$ at $L = 2$. In empty cells, the matrix elements are zero

In general, nonzero elements have the form of

$$a_{0l} = t(1,1,\dots,1,(-1)^l),\ l = 0,1, \tag{9}$$

$$a_{1l} = t(-1,1,\dots,1,(-1)^l)$$

$$a_{2l} = t(1,-1,\dots,1,(-1)^l)$$

$$a_{3l} = t(-1,-1,\dots,1,(-1)^l)$$

$$a_{4l} = t(1,1,\dots,-1,(-1)^l)$$

$$a_{5l} = t(-1,1,\dots,-1,(-1)^l)$$

$$a_{6l} = t(1,-1,\dots,-1,(-1)^l),$$

$$a_{7l} = t(-1,-1,\dots,-1,(-1)^l)$$

Fourier Transform

Let us assume that

$$x = (x_0, x_1, \dots, x_{N-1}), \tag{10}$$

where $N = 2^{L+1}$ - length of sequence x.

Then the Fourier Transform of the sequence x have the form of:

$$X = (X(0), X(1), \dots, X(N-1)) \tag{11}$$

$$X(p) = \sum_{k=0}^{N-1} x_k e^{-\frac{2\pi i}{N}kp},\ p = 0,1,\dots,N-1. \tag{12}$$

Inverse Fourier transform

$$x_j = \frac{1}{N}\sum_{q=0}^{N-1} X(q) e^{\frac{2\pi i}{N}qj} \tag{13}$$

Let us assume that

$$y_k = \sum_{j=0}^{N-1} b_{kj} x_j,\quad k = 0,1,\dots,N-1. \tag{14}$$

Then

$$Y(p) = \sum_{k=0}^{N-1} y_k e^{-\frac{2\pi i}{N}kp} = \frac{1}{N}\sum_{k=0}^{N-1} e^{-\frac{2\pi i}{N}kp} \sum_{j=0}^{N-1} b_{kj} \sum_{q=0}^{N-1} X(q) e^{\frac{2\pi i}{N}qj} = \sum_{q=0}^{N-1}\Big(\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} b_{kj} e^{-\frac{2\pi i}{N}(kp-qj)}\Big) X(q)$$
$$p = 0,1,\dots,N-1. \tag{15}$$

Let us apply the formula (15) to the transfer matrix A:

$$Y(p) = \frac{1}{N}\sum_{m=0}^{1}\sum_{r=0}^{1}\Big(\sum_{k=0}^{N/8-1} a_{4k+r\frac{N}{2},2k+r\frac{N}{4}+m\frac{N}{2}} e^{-\frac{2\pi i}{N}((4k+r\frac{N}{2})p-(2k+r\frac{N}{4}+m\frac{N}{2}))q)} +$$
$$\sum_{k=0}^{N/8-1} a_{4k+1+r\frac{N}{2},2k+r\frac{N}{4}+m\frac{N}{2}} e^{-\frac{2\pi i}{N}((4k+1+r\frac{N}{2})p-(2k+r\frac{N}{4}+m\frac{N}{2}))q)} +$$
$$\sum_{k=0}^{N/8-1} a_{4k+2+r\frac{N}{2},2k+1+r\frac{N}{4}+m\frac{N}{2}} e^{-\frac{2\pi i}{N}((4k+2+r\frac{N}{2})p-(2k+1+r\frac{N}{4}+m\frac{N}{2}))q)} +$$
$$\sum_{k=0}^{N/8-1} a_{4k+3+r\frac{N}{2},2k+1+r\frac{N}{4}+m\frac{N}{2}} e^{-\frac{2\pi i}{N}((4k+3+r\frac{N}{2})p-(2k+1+r\frac{N}{4}+m\frac{N}{2}))q)}\Big)$$
$$p = 0,1,\dots,N-1. \tag{16}$$

Considering that

$$a_{4k+r\frac{N}{2},2k+r\frac{N}{4}+m\frac{N}{2}} = a_{4r,m}$$
$$a_{4k+1+r\frac{N}{2},2k+r\frac{N}{4}+m\frac{N}{2}} = a_{4r+1,m} \tag{17}$$
$$a_{4k+2+r\frac{N}{2},2k+1+r\frac{N}{4}+m\frac{N}{2}} = a_{4r+2,m}$$
$$a_{4k+3+r\frac{N}{2},2k+1+r\frac{N}{4}+m\frac{N}{2}} = a_{4r+3,m}$$

$k = 0,1,\dots,\frac{N}{8}-1,\ \ r = 0,1,\ \ m = 0,1$, we obtain

$$Y(p) = \sum_{q=0}^{N-1} A_{pq} X(q) = \frac{1}{N}\sum_{q=0}^{N-1}\Big(a_{00} + a_{01}e^{\pi iq} + a_{10}e^{-\frac{2\pi i}{N}p} + a_{11}e^{\pi iq}e^{-\frac{2\pi i}{N}p} +$$
$$a_{20}e^{-\frac{2\pi i}{N}(2p-q)} + a_{21}e^{\pi iq}e^{-\frac{2\pi i}{N}(2p-q)} + a_{30}e^{-\frac{2\pi i}{N}(3p-q)} +$$
$$a_{31}e^{\pi iq}e^{-\frac{2\pi i}{N}(3p-q)} + a_{40}e^{-\pi ip+\frac{\pi iq}{2}} +$$
$$+a_{41}e^{-\pi ip+\frac{3\pi iq}{2}} + a_{50}e^{-\pi ip+\frac{\pi iq}{2}-\frac{2\pi i}{N}p} + a_{51}e^{-\pi ip+\frac{3\pi iq}{2}}e^{-\frac{2\pi i}{N}p} +$$
$$a_{60}e^{-\pi ip+\frac{\pi iq}{2}-\frac{2\pi i}{N}(2p-q)} + a_{61}e^{-\pi ip+\frac{3\pi iq}{2}}e^{-\frac{2\pi i}{N}(2p-q)} + a_{70}e^{-\pi ip+\frac{\pi iq}{2}}e^{-\frac{2\pi i}{N}(3p-q)} +$$
$$a_{71}e^{-\pi ip+\frac{3\pi iq}{2}}e^{-\frac{2\pi i}{N}(3p-q)}\Big)S(p,q)X(q),$$
$$\tag{18}$$

where

$$S(p,q) = \sum_{k=0}^{\frac{N}{8}-1} e^{\frac{-2\pi i}{N}(4kp-2kq)} = \frac{1-e^{\frac{-2\pi i}{N}(4p-2q)\frac{N}{8}}}{1-e^{\frac{-2\pi i}{N}(4p-2q)}} = \frac{1-e^{\frac{-\pi i}{2}(2p-q)}}{1-e^{\frac{-2\pi i}{N}(4p-2q)}} \tag{19}$$

$p = 0,1,\dots,N-1$, $2p - q \neq 0 (\bmod \frac{N}{2})$.

Taking advantage of the

$$\frac{1}{1-e^{i\alpha}} = \frac{1}{1-\cos\alpha - i\sin\alpha} = \frac{1}{2\sin^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \tag{20}$$

$$\frac{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2} + i\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}} = \frac{1}{2}+\frac{i}{2}\cot\frac{\alpha}{2}$$

We obtain

$$S(p,q) = \frac{1}{2}(1-(-i)^{2p-q})(1-i\cot(\frac{2\pi}{N}(2p-q))) \tag{21}$$

Let us simplify the expression for matrix elements A_{pq}:

$$A_{pq} = \frac{1}{N}(a_{00} + a_{01}(-1)^q + a_{10}e^{-\frac{2\pi i}{N}p} + a_{11}(-1)^q e^{-\frac{2\pi i}{N}p} +$$
$$a_{20}e^{-\frac{2\pi i}{N}(2p-q)} + a_{21}(-1)^q e^{-\frac{2\pi i}{N}(2p-q)} + a_{30}e^{-\frac{2\pi i}{N}(3p-q)} +$$
$$a_{31}(-1)^q e^{-\frac{2\pi i}{N}(3p-q)} + a_{40}(-1)^p (i)^q + a_{41}(-1)^p (i)^{3q} +$$
$$a_{50}(-1)^p (i)^q e^{-\frac{2\pi i}{N}p} + a_{51}(-1)^p (i)^{3q} e^{-\frac{2\pi i}{N}p} +$$
$$a_{60}(-1)^p (i)^q e^{-\frac{2\pi i}{N}(2p-q)} + a_{61}(-1)^p (i)^{3q} e^{-\frac{2\pi i}{N}(2p-q)} +$$
$$a_{70}(-1)^p (i)^q e^{-\frac{2\pi i}{N}(3p-q)} + a_{71}(-1)^p (i)^{3q} e^{-\frac{2\pi i}{N}(3p-q)})S(p,q) \tag{22}$$

$p = 0,1,\dots,N-1$.

Consider the special case when $a_{l\,0} = a_{l\,1}$, $l = 0,1,\dots,7$. This condition is satisfied by the model when $K_4 = K_5 = K_7 = K_8 = K_9 = 0$, parameters K_1, K_2, K_3, K_6, H they can take arbitrary values, for example, the standard Ising model with an external field and arbitrary interaction parameters in both directions and an additional interaction creating a triangular lattice, plus a triple interaction of the type $\sigma_i\sigma_{i+1}\sigma_{i+L}$ (not including σ_{i+L+1} in terms of this article), "chess-type" Ising model on a triangular lattice. The case of a general Hamiltonian with arbitrary K_4, K_5, K_7, K_8, K_9 the author will consider in the next publication.

In this case from (22), if $q = 2l+1$, $l = 0,1,\dots,\frac{N}{2}-1$, then

$A_{p,2l+1} = 0$, $p = 0,1,\dots,N-1$.

But then it makes sense to take $A_{2m+1,q} = 0$, $m = 0,1,\dots,\frac{N}{2}-1$, $q = 0,1,\dots,N-1$ (still the odd component of the vector $X = \{X(q), q = 0,1,\dots,N-1\}$ when finding a vector corresponding to the maximum eigenvalue, for example, by the power iteration, it will give a zero contribution). As a result, we obtain

$$A_{2m,2l} = 2\frac{1}{N}(a_{00} + a_{10}e^{-\frac{2\pi i}{N}2m} + a_{20}e^{-\frac{2\pi i}{N}2(2m-l)} + a_{30}e^{-\frac{2\pi i}{N}2(3m-l)} +$$
$$a_{40}(-1)^l + a_{50}(-1)^l e^{-\frac{2\pi i}{N}2m} + a_{60}(-1)^l e^{-\frac{2\pi i}{N}2(2m-l)} +$$
$$a_{70}(-1)^l e^{-\frac{2\pi i}{N}2(3m-l)})S(2m,2l) \tag{23}$$

$m = 0,1,\dots,\frac{N}{2}-1$, $l = 0,1,\dots,\frac{N}{2}-1$,

where

$$S(2m,2l) = \frac{1}{2}(1-(-1)^l)(1-i\cot(\frac{2\pi}{N}2(2m-l))) \tag{24}$$

From (24) it is clear that if $l = 2r$, $r = 0,1,\dots,\frac{N}{4}-1$ then $S(2m,4r) = 0$, unless

$2p - q = 4m - 4r \equiv 0$ ($\bmod \frac{N}{2}$),

in this case

$$S(2m,4r) = \frac{N}{8} \tag{25}$$

$\Delta t = 2\frac{2\pi}{N}$, $\Delta\tau = 4\frac{2\pi}{N}$, $t_m = m\Delta t = m\frac{4\pi}{N}$,

$\tau_r = (2r+1)\frac{\Delta\tau}{2} = (2r+1)\frac{4\pi}{N}$.

Then, taking into account (25), (23), we write down

$$\sum_{l=0}^{\frac{N}{2}-1} A_{2m,2l}x_{2l} = \frac{1}{4}(a_{00} + a_{10}e^{-it_m} + a_{20} + a_{30}e^{-it_m} +$$
$$a_{40} + a_{50}e^{-it_m} + a_{60} + a_{70}e^{-it_m})S(2m,2l) \tag{26}$$

Odd columns of the matrix A are:

$$A_{2m,2(2r+1)} = 2\frac{1}{N}(a_{00} + a_{10}e^{-\frac{2\pi i}{N}2m} + a_{20}e^{-\frac{2\pi i}{N}2(2m-(2r+1))} +$$
$$a_{30}e^{-\frac{2\pi i}{N}2(3m-(2r+1))} - a_{40} - a_{50}e^{-\frac{2\pi i}{N}2m} - a_{60}e^{-\frac{2\pi i}{N}2(2m-(2r+1))} -$$
$$a_{70}e^{-\frac{2\pi i}{N}2(3m-(2r+1))})(1-i\cot(\frac{2\pi}{N}2(2m-(2r+1)))) \tag{27}$$

$m = 0,1,\dots,\frac{N}{2}-1$, $r = 0,1,\dots,\frac{N}{4}-1$.

Let us introduce the following $\Delta t = 2\frac{2\pi}{N}$, $\Delta\tau = 4\frac{2\pi}{N}$,

$$t_m = m\Delta t = m\frac{4\pi}{N},\ \tau_r = (2r+1)\frac{\Delta\tau}{2} = (2r+1)\frac{4\pi}{N} \tag{28}$$

Using (27), we represent the right-hand side of (27) in the form

$$A_{2m,2(2r+1)} = \frac{1}{4\pi}\Delta\tau(a_{00} + a_{10}e^{-it_m} + a_{20}e^{-it_m+i\tau_r} + a_{30}e^{-3it_m+i\tau_r} - a_{40} - a_{50}e^{-it_m} - a_{60}e^{-2it_m+i\tau_r} - a_{70}e^{-3it_m+i\tau_r})(1 - i\cot(2t_m - \tau_r)) \tag{29}$$

Using (27) and (29), we pass to the limit when $N \to \infty$. We obtain the form of the operator of the elementary transfer matrix after the Fourier transform

$$\begin{aligned}(Ax)(t) = {} & \frac{1}{4}(a_{00} + a_{10}e^{-it} + a_{20} + a_{30}e^{-it} + a_{40} + a_{50}e^{-it} + \\ & a_{60} + a_{70}e^{-it})x(2t) + \frac{1}{4}(a_{00} + a_{10}e^{-it} - a_{20} - a_{30}e^{-it} + a_{40} + a_{50}e^{-it} - \\ & a_{60} - a_{70}e^{-it})x(2t+\pi) + \\ & \frac{1}{4\pi}[(a_{00} + a_{10}e^{-it} - a_{40} - a_{50}e^{-it})\int_0^{2\pi} x(\tau)d\tau + \\ & (a_{20}e^{-2it} + a_{30}e^{-3it} - a_{60}e^{-2it} - a_{70}e^{-3it})\int_0^{2\pi} e^{i\tau}x(\tau)d\tau - \\ & i((a_{00} + a_{10}e^{-it} - a_{40} - a_{50}e^{-it})\,\mathrm{V.p.}\int_0^{2\pi}\cot(2t-\tau)x(\tau)d\tau + \\ & (a_{20}e^{-2it} + a_{30}e^{-3it} - a_{60}e^{-2it} - a_{70}e^{-3it})\,\mathrm{V.p.}\int_0^{2\pi}\cot(2t-\tau)e^{i\tau}x(\tau)d\tau)]\end{aligned} \tag{30}$$

Conclusive statement

We have obtained an elementary transfer matrix of the generalized Ising model in the form of a sum of integral operators. This will expand the possibilities of studying the spectrum of the transfer matrix and finding the exact or approximate value of the partition function and free energy of the generalized Ising model.

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