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Disorder Solutions for Generalized Ising Model with Multispin Interaction

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Abstract

This study demonstrates a development of convenient formulae for obtaining the value of the free energy in the thermodynamic limit on a set of exact disorder solutions depending on four parameters for a 2D generalized Ising model in an external magnetic field with the interaction of nearest neighbors, next nearest neighbors, all kinds of triple interactions and the four interactions for the planar model, and for the 3D generalized Ising model in an external magnetic field with all kinds of interactions in the tetrahedron formed by four spins: at the origin of the coordinates and the closest to it along three coordinate axes in the first coordinate octant. Lattice models are considered with boundary conditions with a shift (similar to helical ones), and a cyclic closure of the set of all points (in natural ordering). For both the planar model and the 3D model, elementary transfer matrices with non-negative matrix elements are constructed, while the free energy in the thermodynamic limit is equal to the Napierian logarithm of the maximum eigenvalue of the transfer matrix. This maximum eigenvalue can be found for a special kind of eigenvector with positive components. The region of existence of these solutions is described. The examples show the existence of nontrivial solutions of the resulting systems of equations for plane and three-dimensional generalized Ising models. The system of equations and the value of free energy in the thermodynamic limit will remain the same for plane and three-dimensional models with Hamiltonians, in which the value of the maximum in the natural ordering of the spin is replaced by the value of the spin at almost any other point in the lattice, this significantly expands the set of models having disordered exact solutions. The high degree of symmetry and repeatability of the components of the found eigenvectors, disappearing when we go beyond the framework of the obtained set of exact solutions, are the reason for the search for phase transitions in the vicinity of this set of disordered solutions.

Keywords: generalized Ising model, Hamiltonian, multispin interaction, transfer matrix, disorder solutions, partition function, free energy, eigenvector, eigenvalue.

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Неупорядоченные решения обобщенной модели Изинга с мультиспиновым взаимодействием

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Аннотация

В работе получены формулы для нахождения свободной энергии в термодинамическом пределе на множестве точных неупорядоченных решений (disorder solutions), зависящих от четырех параметров для 2D обобщенной модели Изинга во внешнем магнитном поле со взаимодействием ближайших соседей, следующих ближайших соседей (next nearest), всевозможных тройных взаимодействий и взаимодействия четырех спинов для плоской модели, и для 3D обобщенной модели Изинга во внешнем магнитном поле со всевозможными взаимодействиями в тетраэдре, образованном четырьмя спинами: в начале координат и ближайшие к нему по трем координатным осям в первом координатном октанте. Решеточные модели рассматриваются с граничными условиями со сдвигом (похожие на винтовые), и циклическим замыканием множества всех точек (в естественном упорядочении). В обоих случаях для плоской и 3D моделей построены элементарные трансфер-матрицы с неотрицательными матричными элементами, при этом свободная энергия в термодинамическом пределе равна натуральному логарифму максимального собственного значения трансфер-матрицы. Это максимальное собственное значение удается найти для специального вида собственного вектора с положительными компонентами. Описана область существования этих решений. На примерах показано существование нетривиальных решений получающихся систем уравнений для плоских и трехмерных обобщенных моделей Изинга. Система уравнений и значение свободной энергии в термодинамическом пределе останутся прежними для плоских и трехмерных моделей с гамильтонианами, в которых значение максимального в естественном упорядочении спина заменено значением спина практически в любой другой точке решетки, это значительно расширяет множество моделей, имеющих неупорядоченные точные решения. Высокая симметрия и повторяемость компонент найденных собственных векторов, исчезающая при выходе за рамки полученного множества точных решений, является поводом для поиска фазовых переходов в окрестности этого множества неупорядоченных решений.

Ключевые слова: обобщенная модель Изинга, гамильтониан, мультиспиновое взаимодействие, трансфер-матрица, неупорядоченные решения, статистическая сумма, свободная энергия, собственный вектор, собственное значение.

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Introduction

The Ising model with the interaction of pairs of nearest neighbors is one of the most studied statistical mechanics systems. Some exact solutions (with an analytical formula for the partition function or free energy of the system) were obtained mainly for planar models, among these, the exact Onsager solution [1] of the two-dimensional Ising model without an external magnetic field clearly stands out. Publications [2-19] are devoted to the subject of disordered solutions obtained on a subset in the space of parameters of a physical system. Wu F.Y. [2], Baxter [3] provide good reviews on this subject. The transfer matrix apparatus is widely used in statistical physics [1], [10], [20-23]. This article is a logical continuation of the author's work [23], which outlines a general methodology for finding such disordered solutions for generalized Ising and Potts models, here, the author explicitly obtains some such solutions for flat and three-dimensional models of generalized Ising (in particular, formulas are derived for calculating the free energy in the thermodynamic limit for models with Hamiltonians symmetric with respect to the change of signs of all spins), and it is also shown how using the Levenberg -Marquardt method (the Levenberg-Marquardt algorithm) [24] it is possible to obtain a numerical solution of the resulting systems of nonlinear equations.
In this article, we consider generalized Ising models with a general form of multispin interaction with boundary conditions with a shift (similar to helical ones) and cyclic closure of the set of all points (in natural ordering) [23]. For these models, elementary transfer matrices were constructed [23] with non-negative elements, systems of equations were generated and solved, their maximum eigenvalues were obtained (as well as eigenvectors with positive components corresponding to these maximum eigenvalues by the Perron – Frobenius theorem [24]). The Napierian logarithm of the largest eigenvalue is the free energy of the limit system with parameters consistent with the resulting system of equations. An explicit form of the exact solutions of these systems of equations, depending on four parameters, was obtained for sufficiently wide classes, and the region of existence of solutions is also described. The system of equations and the value of free energy in the thermodynamic limit remain the same for plane and for three-dimensional models with Hamiltonians in which the value of the maximum in the natural ordering of the spin is replaced by the value of the spin at almost any other point in the lattice, this significantly expands the set of models with disordered exact solutions (see Note to the section "3D generalized Ising model"). The periodicity of the eigenvectors' components, which disappears when we go beyond the framework of the obtained set of disordered solutions, is a cause for searching phase transitions in the neighborhood of this set.
Now, let us consider ν - dimensional lattice (we accept that $\nu = 2, 3$, (a more detailed description of this lattice can be found in [23], symbols used in article [23] are very similar to our symbols.)
$\mathscr{L}_\nu = \{t = (t_1, t_2, \ldots, t_i, t_{i+1}, \ldots, t_\nu), t_i = 0, 1, \ldots, L_i, i = 1, 2, \ldots, \nu\}$, moreover $(t_1, t_2, \ldots, L_i, t_{i+1}, \ldots, t_\nu) \equiv (t_1, t_2, \ldots, 0, t_{i+1} + 1, \ldots, t_\nu)$,
$i = 1, 2, \ldots, \nu - 1$,

$$(L_1, L_2 - 1, \ldots, L_i - 1, \ldots, L_\nu - 1) \equiv (0, L_2, \ldots, L_i - 1, \ldots, L_\nu - 1) \equiv \ldots \equiv (0, 0, \ldots, L_\nu) \equiv (0, 0, \ldots, 0) \cdot \tag{1}$$

Due to this procedure for identifying points, the lattice $\mathscr{L}_\nu$ has a size of $L_1 \times L_2 \times \ldots \times L_\nu$, total number of lattice nodes $L = L_1 L_2 \ldots L_\nu$. Thus, special boundary cyclic helical (with a shift) conditions are set on the lattice $\mathscr{L}_\nu$. We renumber all points $\mathscr{L}_\nu$:

$$\tau^0 = (0, 0, \ldots, 0), \tau^1 = (1, 0, \ldots, 0), \tau^2 = (2, 0, \ldots, 0), \ldots,$$
$$\tau^{L_1} = (L_1, 0, \ldots, 0) \equiv (0, 1, 0, \ldots, 0), \tau^{L_1+1} = (1, 1, 0, \ldots, 0), \ldots,$$
$$\tau^L = (0, \ldots, 0) \equiv \tau^0 . \tag{2}$$

This numbering determines the natural cyclic traversal of all points (in the positive direction) and the local (cyclic) ordering.
We assume that the is a particle in each node $t = (t_1, t_2, \ldots, t_\nu)$. Particle state is determined by spin σ_t, which at every point of the lattice $t = (t_1, t_2, \ldots, t_\nu)$ can take two values: $\sigma_t \in X = \{+1, -1\}$. Let be $\Omega = \{ t^1, t^2, \ldots, t^p \}$ - some fixed finite subset of points (of a certain form) $\mathscr{L}_\nu$, let us call it the carrier (or the carrier of the Hamiltonian), the lowest point of which $t^{\min} = (0, \ldots, 0)$, the highest one $t^{i_{\max}} = (t_1^{i_{\max}}, t_2^{i_{\max}}, \ldots, t_\nu^{i_{\max}})$ (this does not mean that all points $t^i, i = 0, 1, 2 \ldots, i_{\max}$, belong to | . For example, $\Omega = \{t = (t_1, t_2, \ldots, t_\nu) \in \mathscr{L}_\nu : t_i = 0, 1, i = 1, 2, \ldots, \nu\}$ - unit ν - dimensional cube).
The Hamiltonian of the model can be written as

$$\mathscr{H}(\sigma) = -\sum_{i=0}^{L-1} \sum_{\{ t^1, t^2, \ldots, t^s \} \subseteq \Omega_{\tau^i}} J_{t^1, t^2, \ldots, t^s} \sigma_{t^1} \sigma_{t^2} \ldots \sigma_{t^s} \tag{3}$$

where $\tau^i = (\tau_1^i, \ldots, \tau_\nu^i) \in \mathscr{L}_\nu$, $\Omega_{\tau^i} = \Omega + \tau^i$, $\{ t^1, t^2, \ldots, t^s \} \subseteq \Omega_{\tau^i}$ - arbitrary nonempty subset Ω_{τ^i}, $J_{t^1, t^2, \ldots, t^s}$ - corresponding translation-invariant coefficients of multispin interaction (correspondence with the standard representation of the Hamiltonian can be found in [23]).
We introduce the coefficients $K_{t^1, t^2, \ldots, t^s} = J_{t^1, t^2, \ldots, t^s} / (k_B T)$, where T - temperature, k_B - Boltzmann constant. Then the partition function of the model can be written as following

$$Z_L = \sum_{\{\sigma\}} \exp(-\mathscr{H}(\sigma) / (k_B T)) = \tag{4}$$
$$\sum_{\{\sigma\}} \exp(\sum_{i=0}^{L-1} \sum_{\{ t^1, t^2, \ldots, t^s \} \subseteq \Omega_{\tau^i}} K_{t^1, t^2, \ldots, t^s} \sigma_{t^1} \sigma_{t^2} \ldots \sigma_{t^s})$$

where the summation is performed over all spin states.

Two-Dimensional Generalized Ising Model

Let us concretize the general considerations for a two-dimensional lattice of size $L = L_1 \times L_2$, total number of lattice nodes $L = L_1 L_2$, with special boundary cyclic helical (with shift) conditions (1) and renumbering of points (2). We assume that in each node there is a particle. The particle state is determined by the value (spin) σ_i, which can take 2 values: +1 or -1. Each spin interacts with the eight nearest spins in four directions or lines, $\Omega = \{t = (t_1, t_2) \in \mathscr{L}_2 : t_i = 0, 1, i = 1, 2\}$. The Hamiltonian of the generalized two-dimensional Ising model has the form

$$\begin{aligned} \mathrm{H}(\sigma) = -\sum_{m=1}^{L_2} \sum_{n=1}^{L_1} (& J_1 \sigma_n^m \sigma_{n+1}^m + J_2 \sigma_n^m \sigma_n^{m+1} + \\ & + J_3 \sigma_{n+1}^m \sigma_n^{m+1} + J_4 \sigma_n^m \sigma_{n+1}^{m+1} + J_5 \sigma_n^m \sigma_n^{m+1} \sigma_{n+1}^{m+1} + \\ & + J_6 \sigma_n^m \sigma_{n+1}^m \sigma_n^{m+1} + J_7 \sigma_n^m \sigma_{n+1}^m \sigma_{n+1}^{m+1} + J_8 \sigma_n^{m+1} \sigma_{n+1}^m \sigma_{n+1}^{m+1} + \\ & + J_9 \sigma_n^m \sigma_{n+1}^m \sigma_n^{m+1} \sigma_{n+1}^{m+1} + \hat{h} \sigma_n^m) \end{aligned} \tag{5}$$

where J_i, $i=1,2,...,9$ - corresponding spin-spin interaction coefficients. Let us introduce $K_i = J_i/(k_BT)$, $i=1,2,...,9$, where T - temperature, k_B - Boltzmann constant, $h=\hat{h}/(k_BT)$ interaction parameter with an external field with a coefficient $\hat{h}$. Then the partition function (4) of the model is written as following

$$Z_{L_1L_2}=\sum_{\sigma}\exp(-\mathcal{H}(\sigma)/k_BT)=\sum_{\sigma}\exp(\sum_{m=1}^{L_2}\sum_{n=1}^{L_1}(K_1\sigma_n^m\sigma_{n+1}^m+ +K_2\sigma_n^m\sigma_n^{m+1}+K_3\sigma_{n+1}^m\sigma_n^{m+1}+K_4\sigma_n^m\sigma_{n+1}^{m+1}+K_5\sigma_n^m\sigma_n^{m+1}\sigma_{n+1}^{m+1}+ +K_6\sigma_n^m\sigma_{n+1}^m\sigma_n^{m+1}+K_7\sigma_n^m\sigma_{n+1}^m\sigma_{n+1}^{m+1}+K_8\sigma_n^{m+1}\sigma_{n+1}^m\sigma_{n+1}^{m+1}+ +K_9\sigma_n^m\sigma_{n+1}^m\sigma_n^{m+1}\sigma_{n+1}^{m+1}+h\sigma_n^m)) \quad (6)$$

where the summation is performed over all spin states.
For the model under consideration, we construct an elementary transfer matrix $\mathcal{T}=\mathcal{T}_{p,r}$ in the same way (Fig.1), as in [23], formulas (7-10).

Let us assume that

$$G(\sigma_{\tau^0},\sigma_{\tau^1},\sigma_{\tau^{L_1}},\sigma_{\tau^{L_1+1}},K_1,K_2,...,K_9,h)=\exp(K_1\sigma_{\tau^0}\sigma_{\tau^1}+ +K_2\sigma_{\tau^0}\sigma_{\tau^{L_1}}+K_3\sigma_{\tau^1}\sigma_{\tau^{L_1}}+K_4\sigma_{\tau^0}\sigma_{\tau^{L_1+1}}+K_5\sigma_{\tau^0}\sigma_{\tau^{L_1}}\sigma_{\tau^{L_1+1}}+ +K_6\sigma_{\tau^0}\sigma_{\tau^1}\sigma_{\tau^{L_1}}+K_7\sigma_{\tau^0}\sigma_{\tau^1}\sigma_{\tau^{L_1+1}}+K_8\sigma_{\tau^{L_1}}\sigma_{\tau^1}\sigma_{\tau^{L_1+1}}+ +K_9\sigma_{\tau^0}\sigma_{\tau^1}\sigma_{\tau^{L_1}}\sigma_{\tau^{L_1+1}}+h\sigma_{\tau^0}) \quad (7)$$

				+										-										$\sigma_{\tau^{L_1+1}}$
				+					-					+					-					$\sigma_{\tau^{L_1}}$
						•••					•••					•••					•••			•••
				+	-		+	-	+	-		+	-	+	-		+	-	+	-		+	-	σ_{τ^1}
		+	+	a_{00}		•					•			a_{10}		•					•			
			-	a_{01}										a_{11}										
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			-			•		a_{03}			•					•		a_{13}			•			
		+	+						a_{04}										a_{14}					
			-			•			a_{05}		•					•			a_{15}		•			
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-	⋮			•		•		•		•		•			•		•		•		•		•	
		+	+			•					•	a_{04}				•					•	a_{14}		
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		-	+			•					•		a_{06}			•					•		a_{16}	
			-										a_{07}										a_{17}	
$\sigma_{\tau^{L_1}}$	•••	σ_{τ^1}	σ_{τ^0}																					

F i g. 1. Elementary Transfer Matrix $\mathcal{T}=\mathcal{T}_{p,r}$ for lattice models with Hamiltonian (5) In empty cells there are zero matrix elements

Nonzero matrix elements a_{ij}, $j=0,1,...,7$, $i=0,1$ of elementary transfer matrix $\mathcal{T}=\mathcal{T}_{p,r}$ we write down as follows

$$a_{i0}=G(+1,+1,+1,1-2i,K_1,K_2,...,K_9,h),$$
$$a_{i1}=G(-1,+1,+1,1-2i,K_1,K_2,...,K_9,h)$$
$$a_{i2}=G(+1,-1,+1,1-2i,K_1,K_2,...,K_9,h)$$
$$a_{i3}=G(-1,-1,+1,1-2i,K_1,K_2,...,K_9,h)$$
$$a_{i4}=G(+1,+1,-1,1-2i,K_1,K_2,...,K_9,h)$$
$$a_{i5}=G(-1,+1,-1,1-2i,K_1,K_2,...,K_9,h)$$
$$a_{i6}=G(+1,-1,-1,1-2i,K_1,K_2,...,K_9,h)$$
$$a_{i7}=G(-1,-1,-1,1-2i,K_1,K_2,...,K_9,h) \quad (8)$$

The transfer matrix eigenvector $\mathcal{T}=\mathcal{T}_{p,r}$, corresponding to the highest eigenvalue F, we will seek in the form

$$\vec{x}=(1,b_2,1,b_2,...,1,b_2)^T, \quad (9)$$

where $b_2>0$. Thereafter, by the Perron-Frobenius theorem [25] this eigenvector will correspond to a single maximum eigenvalue F (all matrix elements $\mathcal{T}^n$ for some n will be strictly greater than zero, the full $\mathcal{T}^n$ filled with non-zero elements becomes clear already for $\mathcal{T}^2$. In fact, the Perron-Frobenius theorem is first applied to the matrix $\mathcal{T}^n$). We denote

$$R_i=\exp(K_i),\ i=1,2,...,9,\ H=\exp(h)$$

From the form of the elementary transfer matrix (Fig. 1) and the eigenvector (9), we obtain the following system of equations with

$R_i, i = 1,2,...,9, H, F, b_2$:

$$\begin{cases} F = a_{00} + a_{10} \\ b_2 F = a_{01} + a_{11} \\ F = b_2 a_{02} + b_2 a_{12} \\ b_2 F = b_2 a_{03} + b_2 a_{13} \\ F = a_{04} + a_{14} \\ b_2 F = a_{05} + a_{15} \\ F = b_2 a_{06} + b_2 a_{16} \\ b_2 F = b_2 a_{07} + b_2 a_{17} \end{cases} \quad (10)$$

Or

$$F = \frac{HR_1R_2R_3R_6}{R_4R_5R_7R_8R_9} + HR_1R_2R_3R_4R_5R_6R_7R_8R_9 \quad (11.1)$$

$$b_2F == \frac{R_3R_8}{HR_1R_2R_4R_5R_6R_7R_9} + \frac{R_3R_4R_5R_7R_9}{HR_1R_2R_6R_8} \quad (11.2)$$

$$F = \frac{b_2HR_2R_4R_5}{R_1R_3R_6R_7R_8R_9} + \frac{b_2HR_2R_7R_8R_9}{R_1R_3R_4R_5R_6} \quad (11.3)$$

$$b_2F == \frac{b_2R_1R_4R_5R_6R_8}{HR_2R_3R_7R_9} + \frac{b_2R_1R_6R_7R_9}{HR_2R_3R_4R_5R_8} \quad (11.4)$$

$$F = \frac{HR_1R_4R_7}{R_2R_3R_5R_6R_8R_9} + \frac{HR_1R_5R_8R_9}{R_2R_3R_4R_6R_7} \quad (11.5)$$

$$b_2F == \frac{R_2R_4R_6R_7R_8}{HR_1R_3R_5R_9} + \frac{R_2R_5R_6R_9}{HR_1R_3R_4R_7R_8} \quad (11.6)$$

$$F = \frac{b_2HR_3R_5R_6R_7}{R_1R_2R_4R_8R_9} + \frac{b_2HR_3R_4R_6R_8R_9}{R_1R_2R_5R_7} \quad (11.7)$$

$$b_2F = \frac{b_2R_1R_2R_3R_5R_7R_8}{HR_4R_6R_9} + \frac{b_2R_1R_2R_3R_4R_9}{HR_5R_6R_7R_8} \quad (11.8)$$

Let us solve simultaneous equations (11). The solution scheme is as follows:

1. Derive F from the (11.1),

$$F = \frac{HR_1R_2R_3R_6}{R_4R_5R_7R_8R_9} + HR_11R_2R_3R_4R_5R_6R_7R_8R_9 \quad (12)$$

and substitute it in the remaining equations (11.2)-(11.8).

2. Derive b_2 from the (11.2),

$$b_2 = \frac{R_8^2 + R_4^2R_5^2R_7^2R_9^2}{H^2R_1^2R_2^2R_6^2(1 + R_4^2R_5^2R_7^2R_8^2R_9^2)} \quad (13)$$

and substitute it in the remaining equations (11.3)-(11.8).

3. Derive H^2 from the (11.8)

$$H^2 = \frac{R_5^2R_7^2R_8^2 + R_4^2R_9^2}{R_6^2 + R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2} \quad (14)$$

and substitute it in the remaining equations (11.3)-(11.7).

4. Derive R_6^2 from the (11.5)

$$R_6^2 = \frac{R_4^2R_7^2 + R_5^2R_8^2R_9^2}{R_2^2R_3^2 + R_2^2R_3^2R_4^2R_5^2R_7^2R_8^2R_9^2} \quad (15)$$

and substitute it in the remaining equations (11.3), (11.4), (11.6), (11.7).

5. Derive R_1^4. From the (11.3)

$$R_1^4 = \frac{(R_8^2 + R_4^2R_5^2R_7^2R_9^2)(R_4^2R_5^2 + R_7^2R_8^2R_9^2)}{(R_5^2R_7^2R_8^2 + R_4^2R_9^2)(R_4^2R_7^2 + R_5^2R_8^2R_9^2)} \quad (16)$$

and substitute it in the remaining equations (11.4), (11.6), (11.7).

6. Derive R_3^4 from the (11.6)

$$R_3^4 = \frac{(R_4^2R_7^2R_8^2 + R_5^2R_9^2)(R_4^2R_7^2 + R_5^2R_8^2R_9^2)}{(R_8^2 + R_4^2R_5^2R_7^2R_9^2)(1 + R_4^2R_5^2R_7^2R_8^2R_9^2)} \quad (17)$$

and substitute it in the remaining equations (11.4), (11.7).

7. Derive R_2^4 from the (11.4)

$$R_2^4 = \frac{(R_4^2R_5^2R_8^2 + R_7^2R_9^2)(R_8^2 + R_4^2R_5^2R_7^2R_9^2)}{(R_5^2R_7^2R_8^2 + R_4^2R_9^2)(R_4^2R_7^2R_8^2 + R_5^2R_9^2)} \quad (18)$$

8. Derive R_2^4 from the (11.7)

$$R_2^4 = \frac{(R_5^2R_7^2 + R_4^2R_8^2R_9^2)(R_4^2R_7^2 + R_5^2R_8^2R_9^2)}{(R_4^2R_5^2 + R_7^2R_8^2R_9^2)(1 + R_4^2R_5^2R_7^2R_8^2R_9^2)} \quad (19)$$

9. Equate R_2^4, derived from (18) and (19). We get the quadratic equation $ay^2 + by + c = 0$ (20)

for $y = R_8^2$, where

$$a = R_4^2R_5^2R_7^2R_9^2 + R_4^6R_5^6R_7^2R_9^2 - R_4^6R_5^2R_7^6R_9^2 - \\ -R_4^2R_5^6R_7^6R_9^2 - R_4^6R_5^2R_7^2R_9^6 - R_4^2R_5^6R_7^2R_9^6 + \\ +R_4^2R_5^2R_7^6R_9^6 + R_4^6R_5^6R_7^6R_9^6 \quad (21)$$

$$b = R_4^4R_5^4 - R_4^4R_5^4R_7^8 + R_7^4R_9^4 - R_4^8R_7^4R_9^4 - \\ -R_5^8R_7^4R_9^4 + R_4^8R_5^8R_7^4R_9^4 - R_4^4R_5^4R_9^8 + \quad (22)$$

$$c = R_4^2R_5^2R_7^2R_9^2 + R_4^6R_5^6R_7^2R_9^2 - R_4^6R_5^2R_7^6R_9^2 - \\ -R_4^2R_5^6R_7^6R_9^2 - R_4^6R_5^2R_7^2R_9^6 - R_4^2R_5^6R_7^2R_9^6 + \\ +R_4^2R_5^2R_7^6R_9^6 + R_4^6R_5^6R_7^6R_9^6 \quad (23)$$

Notice, that $a = c$, therefore, by the Vieta theorem, solutions of the equations (20) y_1 and y_2 comply with the formula $y_1 y_2 = 1$, and both roots are of the same sign.

Hence $R_8^2 = \left(-b \pm \sqrt{b^2 - 4ac}\right)/(2a)$,

$$R_8 = R_8(R_4, R_5, R_7, R_9). \qquad (24)$$

Further, in the reverse order by the formulas written above, we find R_2, R_3, R_1, R_6, H, b_2, F. Free energy is $f = \ln(F)$. We get the exact value of free energy $f = f(R_4, R_5, R_7, R_9)$ in the thermodynamic limit on a nonempty set (see examples below) of disordered solutions, depending on four parameters R_4, R_5, R_7, R_9. Conditions for the existence of such disordered solutions:

$$b^2 - 4ac \ge 0, \ \left(-b + \sqrt{b^2 - 4ac}\right)/(2a) > 0. \qquad (25)$$

We transform these conditions to a more convenient form. Let us assume that the discriminant is $D = b^2 - 4ac$ (26)

It can be converted to the following type

$$D = (-R_4^2R_5^2 - R_4^2R_5^2R_7^4 + R_7^2R_9^2 + R_4^4R_7^2R_9^2 + R_5^4R_7^2R_9^2 + R_4^4R_5^4R_7^2R_9^2 - R_4^2R_5^2R_9^4 - R_4^2R_5^2R_7^4R_9^4)$$
$$(-R_4^2R_5^2 + R_4^2R_5^2R_7^4 + R_7^2R_9^2 - R_4^4R_7^2R_9^2 - R_5^4R_7^2R_9^2 + R_4^4R_5^4R_7^2R_9^2 + R_4^2R_5^2R_9^4 - R_4^2R_5^2R_7^4R_9^4)$$
$$(R_4^2R_5^2 - R_4^2R_5^2R_7^4 + R_7^2R_9^2 - R_4^4R_7^2R_9^2 - R_5^4R_7^2R_9^2 + R_4^4R_5^4R_7^2R_9^2 - R_4^2R_5^2R_9^4 + R_4^2R_5^2R_7^4R_9^4)$$
$$(R_4^2R_5^2 + R_4^2R_5^2R_7^4 + R_7^2R_9^2 + R_4^4R_7^2R_9^2 + R_5^4R_7^2R_9^2 + R_4^4R_5^4R_7^2R_9^2 + R_4^2R_5^2R_9^4 + R_4^2R_5^2R_7^4R_9^4) \qquad (27)$$

Or $D = (d_1^2 - d_2^2)(d_3^2 - d_4^2)$, (28)

where

$$d_1 = R_7^2R_9^2 + R_4^4R_7^2R_9^2 + R_5^4R_7^2R_9^2 + R_4^4R_5^4R_7^2R_9^2 \qquad (29)$$

$$d_2 = R_4^2R_5^2 + R_4^2R_5^2R_7^4 + R_4^2R_5^2R_9^4 + R_4^2R_5^2R_7^4R_9^4 \qquad (30)$$

$$d_3 = R_7^2R_9^2 - R_4^4R_7^2R_9^2 - R_5^4R_7^2R_9^2 + R_4^4R_5^4R_7^2R_9^2 \qquad (31)$$

$$d_4 = R_4^2R_5^2R_9^4 - R_4^2R_5^2R_7^4R_9^4 - R_4^2R_5^2 + R_4^2R_5^2R_7^4 \qquad (32)$$

$$D \ge 0 \Leftrightarrow (d_1^2 - d_2^2)(d_3^2 - d_4^2) \ge 0 \Leftrightarrow (|d_1|^2 - |d_2|^2)(|d_3|^2 - |d_4|^2) \ge 0 \Leftrightarrow$$
$$(|d_1| - |d_2|)(|d_3| - |d_4|) \ge 0 \Leftrightarrow (|\cosh(2K_4)\cosh(2K_5)| - |\cosh(2K_7)\cosh(2K_9)|)\times$$
$$(|\sinh(2K_4)\sinh(2K_5)| - |\sinh(2K_7)\sinh(2K_9)|) \ge 0 \qquad (33)$$

Assumptions (25) match (with regard for $a = c$)

$$D \ge 0, \ y_{top} = -\frac{b}{2a} > 0, \text{ или } ab < 0. \qquad (34)$$

After the transformations we get

$$ab < 0 \Leftrightarrow$$
$$(\cosh(2K_4 + 2K_5)\cosh(2K_7 + 2K_9) - \cosh(2K_7 - 2K_9)\cosh(2K_4 + 2K_5))$$
$$(\sinh(4K_7)\sinh(4K_9) + \sinh(4K_4)\sinh(4K_5)) < 0 \qquad (35)$$

Finally, the conditions for the existence of a solution $R_8 = R_8(R_4, R_5, R_7, R_9)$ for parameters $\{K_4, K_5, K_7, K_9\}$:

$$\begin{cases} D \ge 0 \\ ab < 0 \end{cases} \Leftrightarrow \begin{cases} (|\cosh(2K_4)\cosh(2K_5)| - |\cosh(2K_7)\cosh(2K_9)|)\times \\ (|\sinh(2K_4)\sinh(2K_5)| - |\sinh(2K_7)\sinh(2K_9)|) \ge 0 \\ (\cosh(2K_4 + 2K_5)\cosh(2K_7 + 2K_9) - \\ \cosh(2K_7 - 2K_9)\cosh(2K_4 + 2K_5))\times \\ (\sinh(4K_7)\sinh(4K_9) + \sinh(4K_4)\sinh(4K_5)) < 0 \end{cases} \qquad (36)$$

Examples

1. Let us assume that $R_8^2 = \left(-b + \sqrt{b^2 - 4ac}\right)/(2a)$.

$R_9 = 1.1$, $R_7 = 8$, $R_4 = 2$, $R_5 = 0.5$ (R_9, R_7, R_4, R_5 - these are arbitrary parameters chosen by us, other values are calculated by the formulas), $R_8 = 0.5008038793422166$

$R_2 = 1.8030575636447406$,
$R_3 = 1.7962953635956354$,
$R_1 = 0.9146508996595731$,
$R_6 = 1.0933133071559797$,
$H = 0.6022060529968286$,
$b_2 = 3.2266770332751755$,
$F = 9.038301092103787$,
$f = \ln F = 2.2014712244531953189$.

2. Let us assume that

$R_8^2 = \left(-b - \sqrt{b^2 - 4ac}\right)/(2a)$.

$R_9 = 1.1$, $R_7 = 8$, $R_4 = 2$, $R_5 = 0.5$,
$R_8 = 1.9967896441086979$
$R_2 = 0.5546134633542025$,
$R_3 = 1.7962953635956356$,
$R_1 = 1.0933133071559797$,
$R_6 = 0.9146508996595731$,
$H = 0.5146350790279501$,
$b_2 = 3.2266770332751755$,
$F = 9.038301092103787$,
$f = \ln F = 2.2014712244531953189$.

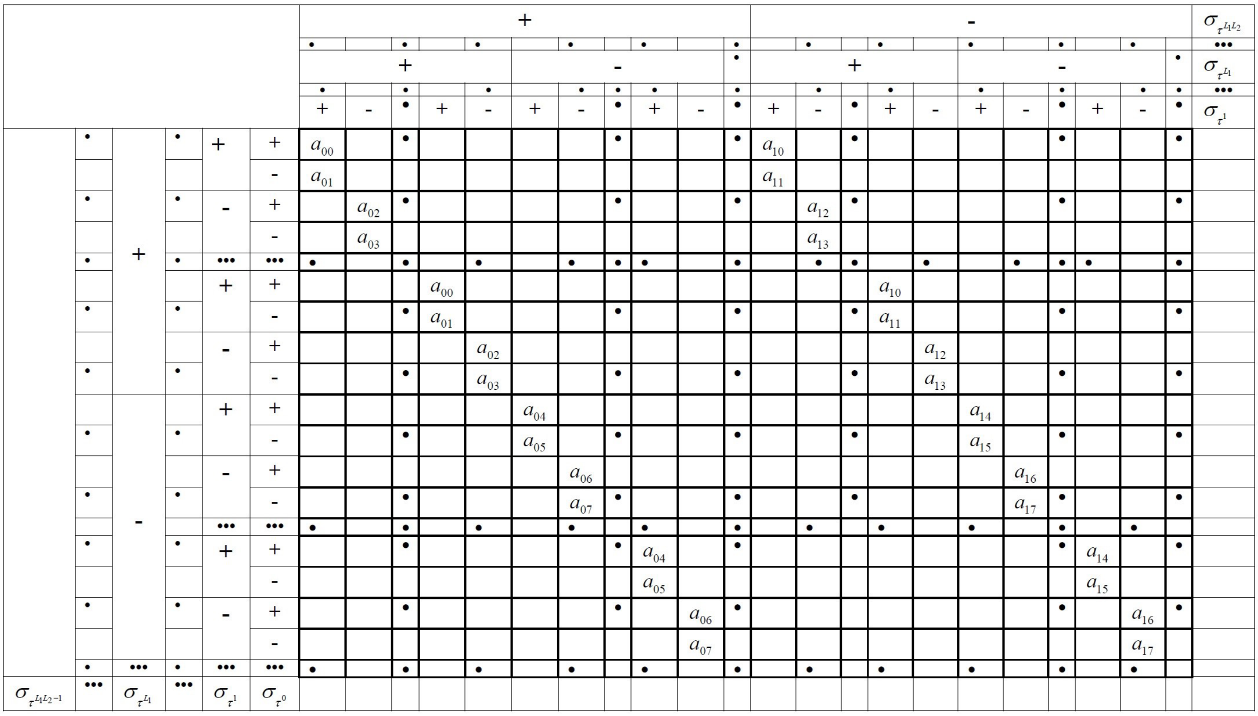
In examples 1 and 2, the initial values of the parameters are the same; two branches of the solution of the quadratic equation (20) are considered. We see that $R_3 = 1.7962953635956356$ the same thing in both examples, R_1 and R_6 swap, b_2 and F are the same.

3D Generalized Ising Model

Let us consider 3D Generalized Ising Model размера $L = L_1 \times L_2 \times L_3$, total number of grid nodes $L = L_1L_2L_3$, with special boundary cyclic helical (with shift) conditions (1) and renumbering of points (2). Let us assume that $\Omega = \{\tau_0, \tau_1, \tau_{L_1}, \tau_{L_1L_2}\} = \{t_0, t_1, t_2, t_3\}$. Hamiltonian of 3D models is

$$\mathrm{H}(\sigma) = -\sum_{i=1}^{L_2}\sum_{n=1}^{L_1}(J_{01}\sigma_0\sigma_1 + J_{02}\sigma_0\sigma_2 + J_{03}\sigma_0\sigma_3 +$$
$$+J_{12}\sigma_1\sigma_2 + J_{13}\sigma_1\sigma_3 + J_{23}\sigma_2\sigma_3 + J_{012}\sigma_0\sigma_1\sigma_2 + J_{013}\sigma_0\sigma_1\sigma_3 +$$
$$+J_{023}\sigma_0\sigma_2\sigma_3 + J_{123}\sigma_1\sigma_2\sigma_3 + J_{0123}\sigma_0\sigma_1\sigma_2\sigma_3 + h_0\sigma_0 \qquad (37)$$

Let us introduce the following transfer matrix $\mathcal{T} = \mathcal{T}_{p,r}$ for this model, we will do it the same way as in [23] (Fig. 2).



F i g. 2. Elementary Transfer Matrix $\mathscr{T} = \mathscr{T}_{p,r}$ for lattice models with Hamiltonian (37). In empty cells there are zero matrix elements

Eigenvector of the transfer matrix $\mathscr{T} = \mathscr{T}_{p,r}$, corresponding to the highest eigenvalue F, will be searched in the form of (9). There is a one-to-one correspondence between the system of equations generated by the Hamiltonian (37) and the system of equations (10) - (11) for the planar model generated by the Hamiltonian (5), this is clear from the correspondence of the vertices: $(n,m) \leftrightarrow \tau_0$, $(n+1,m) \leftrightarrow \tau_1$, $(n,m+1) \leftrightarrow \tau_{L_1}$, $(n+1,m+1) \leftrightarrow \tau_{L_1L_2}$. That is, for a three-dimensional model with Hamiltonian (37) there will be the same system of equations for the interaction coefficients and free energy in the thermodynamic limit as for a flat model with interaction coefficients

$$J_1 = J_{01} + J_{23},\ J_2 = J_{02} + J_{13},$$

$$J_3 = J_{12},\ J_4 = J_{03},\ J_5 = J_{023},\ J_6 = J_{012},\ J_7 = J_{013},\ J_8 = J_{123},\ J_9 = J_{0123},\ \hat{h} = h_0 .$$

Comment

In a flat generalized Ising model $t_3 = \tau_{L_1+1}$, in a 3D generalized Ising model $t_3 = \tau_{L_1L_2}$, however, the system of equations (10) - (11) and the value of free energy f will remain the same if we take any point for the 3D generalized Ising model (and, consequently, $\sigma_{t_3} \equiv \sigma_3$ в (37)), $\tau_{L_1+1} < t_3 < \tau_{L_1L_2L_3}$, and for a flat model $\tau_{L_1+1} < t_3 < \tau_{L_1L_2}$ (and, consequently, σ_{t_3} instead of σ_{n+1}^{m+1} в (5)), which significantly expands the class of models with Hamiltonians for which exact disorder solutions of the form (12-24) are found.

Example 3. (it is induced by example 1 for a flat model).

$$K_{01} + K_{23} = K_1 = \ln(0.9146508996595731),$$
$$K_{02} + K_{13} = K_2 = \ln(1.8030575636447406),$$
$$K_{12} = K_3 = \ln(1.7962953635956354),$$
$$K_{03} = K_4 = \ln(2),\ K_{023} = K_5 = \ln(0.5),$$
$$K_{012} = K_6 = \ln(1.0933133071559797),$$
$$K_{013} = K_7 = \ln(8),$$
$$K_{123} = K_8 = \ln(0.5008038793422166),$$
$$K_{0123} = K_9 = \ln(1.1),$$
$$h_0 = \ln(0.6022060529968286),$$
$$b_2 = 3.2266770332751755,$$
$$F = 9.038301092103787,$$
$$f = \ln F = 2.2014712244531953189 .$$

Conclusive statement

This study demonstrates obtaining the value of the free energy in the thermodynamic limit on a set of exact disorder solutions depending on four parameters for a 2D generalized Ising model in an external magnetic field with the interaction of nearest neighbors, next nearest neighbors, all kinds of triple interactions and the four interactions for the planar model, and for the 3D generalized Ising model in an external magnetic field with all kinds of interactions in the tetrahedron formed by four spins: at the origin of the coordinates and the closest to it along three coordinate axes in the first coordinate octant. The domain of existence of these solutions is also described. The system of equations (10) - (11) and the value of free energy f remain the same if we take any point for the 3D generalized Ising model t_3 (and accordingly $\sigma_{t_3} \equiv \sigma_3$ in (37)), $\tau_{L_1+1} < t_3 < \tau_{L_1L_2L_3}$, and for a flat model $\tau_{L_1+1} < t_3 < \tau_{L_1L_2}$ (and accordingly σ_{t_3} instead of σ_{n+1}^{m+1} in (5)), which significantly extends the class of models with Hamiltonians for which exact disordered solutions of the form (12-24) are found.

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