УДК 519.6:517.51
DOI: 10.25559/SITITO.16.202001.33-40

# On Conservative Averaging Method in Spline Applications

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## Abstract

We consider the conservative averaging method for solving the 3-D boundary-value problem of second order in multilayer domain. Looking back to the history of mathematics, integral parabolic splines relates to conservative averaging method (CAM) introduced by A. Kneser in 1914. In 1980's, A. Buikis had developed CAM method for partial differential equations with discontinuous coefficients, when he was modelling processes in environments with a layered structure. The special hyperbolic and exponential type splines, with middle integral values of piecewise smooth function interpolation, are considered. Using these type splines, the problems of mathematical physics in 3-D with piecewise coefficients are reduced to 2-D problems with respect to one coordinate. This procedure also allows reducing the 2-D problems to 1-D problems and the solution of the approximated problems can be obtained analytically. In the case of constant piecewise coefficients, we obtain the exact discrete approximation of a steady-state 1-D boundary-value problem. Similarly, the approximation of the 3-D nonstationary problem is obtained with CAM. The numerical solution is compared with the analytical solution.

**Keywords:** special splines, averaging method, 3D problem, analytical solution.

**For citation:** Kalis H., Kangro I. On Conservative Averaging Method in Spline Applications. *Sovremennye informacionnye tehnologii i IT-obrazovanie* = Modern Information Technologies and IT-Education. 2020; 16(1):33-40. DOI: https://doi.org/10.25559/SITITO.16.202001.33-40

# О консервативном методе усреднения в сплайновых приложениях

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## Аннотация

В статье рассматривается консервативный метод усреднения для решения трехмерной краевой задачи второго порядка в многослойной области. Оглядываясь назад на историю математики, мы видим, что интегральные параболические сплайны относятся к консервативному методу усреднения (САМ), введенному А. Кнезером в 1914 году. В 1980-х годах А. Буйкис разработал метод САМ для уравнений в частных производных с разрывными коэффициентами, когда он моделировал процессы в средах со слоистой структурой. Рассматриваются специальные сплайны гиперболического и экспоненциального типов со средними интегральными значениями интерполяции кусочно-гладкой функции. Используя сплайны такого типа, задачи математической физики в трехмерном пространстве с кусочными коэффициентами сводятся к двумерным задачам относительно одной координаты. Эта процедура также позволяет свести двумерные задачи к одномерным задачам, и решение аппроксимированных задач может быть получено аналитически. В случае постоянных кусочных коэффициентов мы получаем точную дискретную аппроксимацию стационарной 1-й краевой задачи. Аналогично аппроксимация трехмерной нестационарной задачи получается с помощью САМ. Численное решение сравнивается с аналитическим решением.

**Ключевые слова:** специальные сплайны, метод усреднения, трехмерная задача, аналитическое решение.

**Для цитирования:** Калис, Х. О консервативном методе усреднения в сплайновых приложениях / Х. Калис, И. Кангро. – DOI 10.25559/SITITO.16.202001.33-40 // Современные информационные технологии и ИТ-образование. – 2020. – Т. 16, № 1. – С. 33-40.

## Introduction

The paper is devoted to the memory of Latvian mathematician E. Grinbergs (1911-1982), pointed out his outstanding achievements in applied mathematics, namely: radio filters, hulls of tankers, graphs theory and integral circuits, especially his work in the construction of the tanker fleet of the USSR [1]. According to his calculations of tanker hulls, in the period 1963-1970, 23 veru large tankers were built in Leningrad. At that time the research was secret, and it was not reported in the open sources. A summary of the calculation methodology (without any reference to the application objects) contains the article [2]. Today this interpolation methodology relates to the mathematical methods of splins.

The term "spline" is used to refer to a wide class of functions that are used in applications requiring data interpolation and/or smoothing. The data may be either one-dimensional or multi-dimensional. A spline function is a piecewise polynomial function. The places where the pieces meet are named knots. The key property of spline functions is that they and their derivatives may be continuous, depending on the multiplicities of the knots.

Integral parabolic splines (IPS) for the first time were defined by A.Buikis in [3] and developed in his doctoral dissertation [4]. Forty years ago, it was important to construct mathematical models for intensification of the crude oil or gas output. For the situation with layered media, integral splines were introduced to consider the energy or mass conservation in new simplified (less dimensional) formulation of the problem.

Looking back to the history of mathematics, integral parabolic splines relates to conservative averaging method (CAM) introduced in 1914 by A.Kneser [5]. In 1980's, A. Buikis had developed CAM method for partial differential equations with discontinuous coefficients, when he was modelling processes in environments with a layered structure. The conservative averaging method was developed as approximate analytical and numerical method for solving partial differential equations (PDE) with piecewise continuous coefficients. To apply this method for the layered media, a special type of quadratic spline was developed, namely: the integral averaged values interpolating quadratic spline.

Later, the concept of integral spline was generalized by defining other types of splines besides polynomials, namely: hyperbolic and exponential type splines [6] considered below in this paper.

## Integral quadric spline

Given a continuous, smooth piece function $U(x)$, $x \in [a, b]$. It is assumed that the first derivatives at the inner points $t_i$ of $U(x)$ have a final jump: $K_{i-1}\, U'(t_i-0) = K_i\, U'(t_i+0)$, $i=1,\dots N$, where $K_i$ are given positive coefficients. The continuity executes the equations: $U(t_i-0) = U(t_i+0)$, $i=1,\dots N$. The mean integral values for $U(x)$ of $u_i$ by subsegment $[t_i, t_{i+1}]$ are given

$$u_i=\frac{1}{H_i}\int_{t_i}^{t_{i+1}} U(x)dx,\quad H_i = t_{i+1} - t_i,\ i=0\dots N,\ t_0=a,\ t_{N+1}=b.$$

It is necessary to approximate the function $U(x)$ using the previous conditions and the following general boundary conditions (BC) at the interval endpoints $x=a$ and $x=b$:

$$K_0U'(a)- \alpha(U(a)-F_0)=0,$$

$$K_NU'(b)+ \beta(U(b)-F_1)=0,$$

where $\alpha$, $\beta$ are positive coefficients, $F_1$, $F_2$ - given constants. Such BCs are typical one for the ordinary and partial differential equations (the third kind or so called Robin BCs; the second kind or so called Neumann BCs (if $\alpha=\beta=0$); the first kind, or Dirichlet BCs (if $\alpha=\beta=\infty$) and periodical BCs in case: $U(a)=U(b)$, $U'(a)=U'(b)$). The proof is given that this interpolation problem can be solved by a second order interpolation polynomial spline, in other words, by the integral quadric spline of the form:

$$S(x) = u_i + m_i\,(x-\bar{t}_i) + e_i\, G_i\left(\frac{(x-\bar{t}_i)^2}{H_i^2} - \frac{1}{12}\right),$$

where $\bar{t}_i=(t_i+t_{i+1})/2$, $G_i=H_i/K_i>0$.

From the previous conditions the 2(1+ N) unknown coefficients $m_i$, $e_i$ can be determined. The mean integral values of $u_i$ can be also determined by averaging the differential equations.

## Hyperbolic and exponential type splines for 1-D stationary problem in N layered domains

In [6], we consider averaging methods for solving the 3-D boundary value problem in domain containing peat blocks. We consider the metal concentration in the peat block. A specific feature of these problems is that it is necessary to solve the 3-D boundary value problems for elliptic type partial differential equation of the second order (the diffusion equation) with piecewise diffusion coefficients in every direction. The special hyperbolic and exponential type splines, with interpolation of middle integral values by piecewise smooth function, are defined. This procedure allows us to transform the 3-D problem to 2-D and 1-D problems and the solution of the approximated problem is obtained analytically [7]. The numerical solution is compared below with the analytical solution.

Firstly, we consider the 1-D diffusion and diffusion-convections boundary value problems in $N$ layered domains.

**Example 1.** We considered the following 1-D diffusion problem in $[0, L]$ for N layers:

1) differential equations:

$$D_i\frac{\partial^2}{\partial z^2}u_i(z) - a_i^2u_i(z) + F_i = 0,\ z\in [z_{i-1}, z_i],\ i=\overline{1,N},\ z_0=0,\ z_N = L,$$

2) third kind BCs for $z=0$:

$$D_1\frac{\partial u_1(0)}{\partial z} - \alpha\,(u_1(0)-u_0)=0,\ \text{and for } z=L:\ D_N\frac{\partial u_N(L)}{\partial z} + \beta\,(u_N(L) - u_L)=0,$$

3) continuous conditions:

$$u_i(z_i)=u_{i+1}(z_i),\ D_i\frac{\partial u_i(z_i)}{\partial z} = D_{i+1}\frac{\partial u_{i+1}(z_i)}{\partial z},\ i=1,\dots,N-1,$$

where $D_i$ are constant diffusion coefficients, $\alpha$, $\beta$ - mass transfer coefficients, $a_i$, $F_i$, $u_0$, $u_L$ – fixed constant values.

For this problem we can obtain the analytical solution in the following form:

$$u_i(z)=m_i\sinh(a_i^*(z-z_i^*)) + e_i\cosh(a_i^*(z-z_i^*))+F_i/a_i^2,$$

$$z_i^*=(z_{i-1}+z_i)/2,\ a_i^*= \frac{a_i}{\sqrt{D_i}}.$$

From BCs and the continuous conditions, we obtain $N$ linear algebraic equations with tridiagonal matrix for determination of the unknown coefficients $e_i$ and then $m_i$, using integral hyperbolic type splines:

$$u_i(z)= u_{iz} + m_{iz}\frac{0.5\,H_i\sinh(a_{i1}\,(z-z_{i*}))}{\sinh(0.5\,a_{i1}H_i\}} + e_{iz}\frac{\cosh(a_{i1}\,(z-z_{i*}))-A_i}{8\sinh^2(0.25\,a_{i1}H_i\}} \quad (1)$$

$$A_i=\frac{\sinh(0.5a_{i1}H_i)}{0.5a_{i1}H_i}$$

where $u_{iz}=\frac{1}{H_i}\int_{z_{i-1}}^{z_i} u_i(z)dz$ are the mean integral values of $u_i(z)$, $H_i=z_i - z_{i-1}$, $a_{i1}= a_i* = \frac{a_i}{\sqrt{D_i}}$.

Similarly, from BCs and the continuous conditions we can determine the unknown coefficients $e_{iz}$ and $m_{iz}$ depending on the mean integral values $u_{iz}$. By using the mean integral values of differential equation for determination of $u_{iz}$, we obtain linear algebraic equation in following form:

$0.5\, D_i e_{iz} a_{i1} coth(0.25\, a_{i1}H_i)/H_i - a_i^2 u_{iz} + F_i = 0.$

In limit case when the parameters $a_{i1}$ for the hyperbolic spline (1) tends to zero, we have the integral parabolic spline, obtained from A. Buikis [3], [4]. We proved that the hyperbolic type splines give exact solution for the previous boundary-value problem with $m_i = 0.5 m_{iz} H_i / sinh(0.5\, a_{i1}H_i)$,

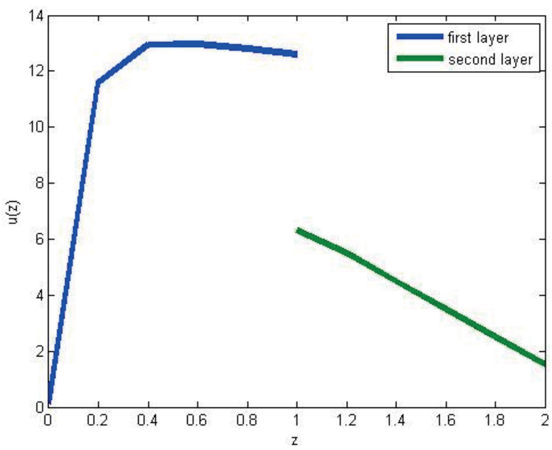
$e_i = 0.125 e_{iz}/sinh^2(0.25\, a_{i1}H_i)$,
$F_i/a_i^2 = u_{iz} - e_{iz}\frac{sinh(0.5\, a_{i1}H_i)}{4\, sinh^2(0.25\, a_{i1}H_i) a_{i1}H_i} = u_{iz} - e_i A_i.$

**Example 2.** We consider following equations for 1-D diffusion - convection problem in $[0,L]$ for $N$ layers with continuous conditions and other BCs:

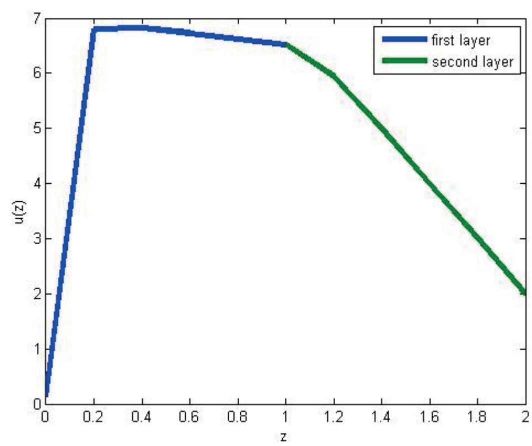
$D_i\frac{\partial^2}{\partial z^2}u_i(z) + r_i\frac{\partial}{\partial z}u_i(z) - a_i^2 u_i(z) + F_i = 0,$
$z \in [z_{i-1}, z_i]$, $i=\overline{1,N}$, $z_0=0$, $z_N=L$,
where $r_i$ is a constant convection velocity in every layer. For this problem we can obtain the analytical solution in the following form:
$u_i(z) = m_i\, exp\,(a_{i-}(z - z_i^*)) + exp\,(a_{i+}(z - z_i^*)) + F_i/a_i^2,$
$z_i^* = (z_{i-1} + z_i)/2,$
$a_{i-} = -\frac{r_i}{2D_i} - \sqrt{\frac{r_i^2}{4D_i^2} + \frac{a_i^2}{D_i}},\ a_{i+} = -\frac{r_i}{2D_i} + \sqrt{\frac{r_i^2}{4D_i^2} + \frac{a_i^2}{D_i}}.$
From BCs at $z=0$ and $z=L$ and the continuous conditions:
$r_i\, u_i(z_i) = r_{i+1}\, u_{i+1}(z_i)$, $D_i\frac{\partial u_i(z_i)}{\partial z} = D_{i+1}\frac{\partial u_{i+1}(z_i)}{\partial z}$, $i=1,\ldots,N-1$,
we obtain $2N$ linear algebraic equations to determine the unknown coefficients $e_i$ and $m_i$.
Using integral exponential type splines, we get solution:
$u_i(z) = u_{iz} + m_{iz}\,(exp\,(a_{i-}(z - z_i^*)) - q_{i-}) + e_{iz}\,(exp\,(a_{i+}(z - z_i^*)) - q_{i+}),$ (2)
where $q_{i-} = \frac{2}{a_{i-}H_i} sinh\frac{a_{i-}H_i}{2}, q_{i+} = \frac{2}{a_{i+}H_i} sinh\frac{a_{i+}H_i}{2}$.
Similarly, from BCs at $z=0$ and $z=L$ and the continuous conditions, we determine the unknown coefficients $e_{iz}$ and $m_{iz}$ depending on the mean integral values $u_{iz}$. By using the mean integral values of differential equation for determination of $u_{iz}$, we obtain a linear algebraic equation in the following form:
$2m_{iz} sinh(0.5\ a_{i-}H_i)(D_i a_{i-} + r_i) + 2e_{iz} sinh(0.5\ a_{i+}H_i)(D_i a_{i+} + r_i) - H_i a_i^2 u_{iz} + H_i F_i = 0.$
Thus, we can see that the exponential type splines give the exact solution with
$m_i = m_{iz}$, $e_i = e_{iz}$ and $F_i/a_i^2 = u_{iz} - e_{iz} q_{i+} - m_{iz} q_{i-}$.

**Example 3.** We considered following equations for simple ($a_i=0$) 1-D diffusion-convection problem in $[0, L]$ for $N$ layers with continuous conditions and BCs:
$D_i\frac{\partial^2}{\partial z^2}u_i(z) + r_i\frac{\partial}{\partial z}u_i(z) + F_i = 0, z \in [z_{i-1}, z_i],$
$i=\overline{1,N}$, $z_0=0$, $z_N=L$, (3)
where $r_i \neq 0$ is a constant convection velocity in every layer. For this problem we can obtain the analytical solution in following form:
$u_i(z) = m_i\,(z - z_i^*) + e_i\, exp\,(a_{i1}(z - z_i^*)) + C_i,$
$a_{i1} = -r_i/D_i$, $m_i = -F_i/r_i$, $z_i^* = (z_{i-1} + z_i)/2.$
From BCs at $z=0$ and $z=L$ and the continuous conditions:
$r_i\, u_i(z_i) = r_{i+1}\, u_{i+1}(z_i)$,
$D_i\frac{\partial u_i(z_i)}{\partial z} = D_{i+1}\frac{\partial u_{i+1}(z_i)}{\partial z}$, $i=1,\ldots,N-1$
we obtain $2N$ linear algebraic equations for determination of the unknown coefficients $e_i$ and $C_i$.
By using the integral exponential type splines, we get:
$u_i(z) = u_{iz} + m_{iz}(z - z_i^*) + e_{iz}(exp\,(a_{i1}(z - z_i^*)) - q_i),$ (4)
where $q_i = \frac{2}{a_{i1}H_i} sinh\frac{a_{i1}H_i}{2}$, $m_{iz} = -F_i/r_i$.
Similarly, from BCs at $z=0$ and $z=L$ and the continuous conditions, we determine the unknown coefficients $e_{iz}$ and $u_{iz}$. We can see that using the mean integral values of differential equation we obtain the following identity:
$2e_{iz} a_{i1} sinh(0.5\, a_{i1}H_i)/H_i - a_{i1}(m_{iz}H_i + e_{iz}\, 2e_{iz}\, sinh(0.5\, a_{i1}H_i))/H_i + F_i/D_i = 0.$
We can see, that this exponential type splines gives exact solution with $m_i = m_{iz}$, $e_i = e_{iz}$ and $C_i = u_{iz} - e_{iz} q_i$.

**A particular case.** For the given parameters: $N=L=2$, $H_1=H_2=1$, $r_1=1$, $r_2=2$, $F_1=1$, $F_2=10$, $\alpha=100$, $\beta=1$, $u_0=0$, $u_L=1$, $D_1=0.01$, $D_2=0.01$ we have discontinuous solutions: $u_1(1)=12.60$, $u_2(1)=6.30$, $u_{1z}=11.75$, $u_{2z}=3.99$ (Fig.1.)
If $r_1 = r_2 = 2$ and $D_1=0.01$, $D_2=0.02$ then we have continuous solutions and discontinuous derivatives, having the numerical values $u_1(1)=u_2(1)=6.525$, $u_{1z}=6.43$, $u_{2z}=4.45$ (Fig.2.)



F i g.1. The solutions of two layers for $r_1=1$, $r_2=2$, $D_1=D_2=0.01$



F i g. 2. The solutions of two layers for $r_1=2$, $r_2=2$, $D_1=0.01$, $D_2=0.02$

**Example 4.** Using the transformation $u_i(z)= exp(-r_{i*}(z-z_{i*})\, v_i(z)$ for equation (3) in every layer we obtain the following problem:

$$\frac{\partial^2}{\partial z^2} v_i(z) - r_{i*}^2 v_i(z) + F_{i*} exp(r_{i*}(z - z_{i*}) = 0, \qquad (5)$$

$z\in [z_{i-1}, z_i]$, $i=\overline{1,N}$, $z_0=0$, $z_N=L$

$D_1 \frac{\partial v_1(0)}{\partial z} - (\alpha + r_{1*}) v_1(0) + \alpha u_0 = 0$, (BCs for $z=0$),

$D_N \frac{\partial v_N(L)}{\partial z} + (\beta - r_{N*}) v_N(L) - \beta u_L exp(r_{N*}L) = 0$, (BCs for $z=L$),

$r_i v_i(H_i) = r_{i+1} v_{i+1}(H_i) \exp((r_{i*} - r_{i+1*})H_i)$,

$D_i \frac{\partial v_i(H_i)}{\partial z} = D_{i+1} \frac{\partial v_{i+1}(H_i)}{\partial z} \exp((r_{i*} - r_{i+1*})H_i)$,

$i=1,\ldots,N-1$, (continuous conditions),

$F_{i*} = F_i / D_i, r_{i*} = r_i/(2 D_i)$.

For this problem, we can obtain the analytical solution in the following form:

$v_i(z) = m_i \ sinh(a_i^* (z - z_{i*})) + e_i cosh(a_i^* (z - z_{i*})) + g_i(z)$,

$z_{i*} = (z_{i-1} + z_i)/2, a_i^* = r_{i*}$,

where $g_i(z) = F_{i*} \ exp(r_{i*}(z - z_{i*})/(4r_{i*}^2)(1 - 2r_{i*}(z - z_{i*}))$ is the particular solution of (5).

From BCs and the continuous conditions, one can obtain $N$ linear algebraic equations for the determination of unknown coefficients $e_i$ and $m_i$, using integral hyperbolic type splines (1) in the following form:

$v_i(z) = v_{iz} + m_{iz} \frac{0.5 H_i sinh(a_{i1}(z - z_{i*}))}{sinh(0.5\, a_{i1} H_i)} + e_{iz} \frac{cosh(a_{i1}(z - z_{i*})) - A_i}{8\, sinh^2(0.25\, a_{i1} H_i)}$,

$A_i = \frac{sinh(0.5 a_{i1} H_i)}{0.5 a_{i1} H_i}$,

where $v_{iz} = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} v_i(z)dz$ are the mean integral values of $v_i(z)$, $a_{i1} = r_{i*}$

Similarly, from BCs and the continuous conditions we can determine the unknown coefficients $e_{iz}$ and $m_{iz}$ depending on the mean integral values $v_{iz}$. By use of the mean integral values of differential equation, we obtain $N$ linear algebraic equations to determine $v_{iz}$:

$0.5\, e_{iz} a_{i1} coth(0.25\, a_{i1} H_i)/H_i - r_{i*}^2 v_{iz} + 2F_{i*}\, sinh(0.5\, r_{i*} H_i)/(r_{i*} H_i) = 0$.

We can see, that the hyperbolic type splines give exact solution for the previous boundary-value problem in case of $F_i=0$, then $m_i=0.5 m_{iz} H_i/sinh(0.5\, a_{i1} H_i)$, $e_i=0.125 e_{iz}/sinh^2(0.25\, a_{i1} H_i)$, $v_{iz} = e_i A_i$.

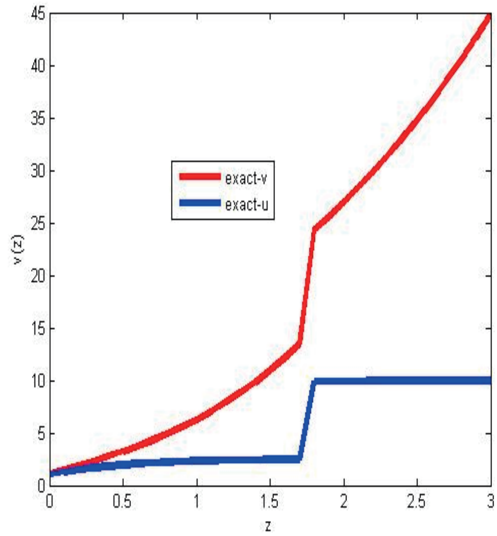
In case of $F_i$ nonzeros and $g_{iz} = v_{iz} - e_i A_i$, we obtain the approximate solution

$g_{iz} = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} g_i(z)dz = F_{i*}/(4H_i r_{i*}^2)((2/r_{i*} - 2H_i) sinh(0.5\, r_{i*} H_i) + 4cosh(0.5\, r_{i*} H_i))$

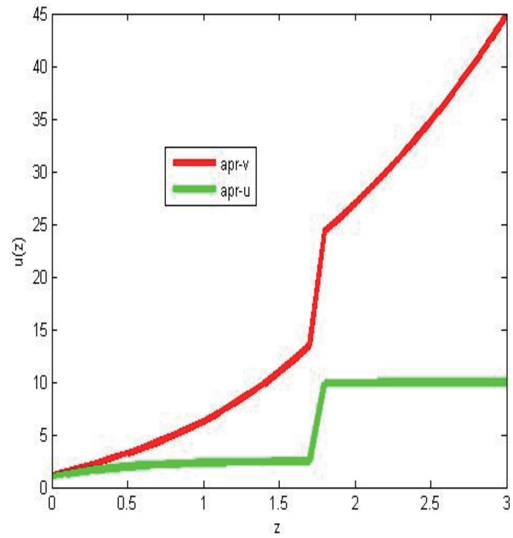
**A particular case.** For $N=2$, $L_z=3$, $H_1=1.8$, $H_2=1.2$, $r_1=0.4$, $r_2=0.1$, $F_1=0$, $\alpha=100, \beta=1000$, $u_0=1, u_L=10$, $D_1=0.2$, $D_2=0.1$ we have discontinuous solutions $u_1(H_1) - u_2(H_1) = -7.413$, $v_1(H_1) - v_2(H_1) = -9.362$, $v_{1z}=6.347$, $v_{2z}=33.560$ (Figs 3 and 4).

If $r_1 = r_2 = 0.1$ and $D_1=0.5$, $D_2=0.1$ then we have the following u-continuous solutions, v-discontinuous solutions and discontinuous u-derivatives:

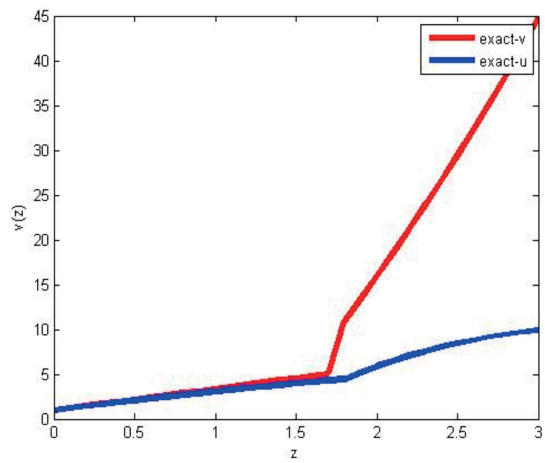
$u_1(H_1) - u_2(H_1) = 0$, $v_1(H_1) - v_2(H_1) = -5.605$, $v_{1z}=3.150$, $v_{2z}=27.036$ (Fig.5.)




F i g. 3. The exact v- and u-solutions for two layers: $r_1=0.4$, $r_2=0.1$, $D_1=0.2$, $D_2=0.1$




F i g. 4. The spline v- and u-solutions for two layers: $r_1=0.4$, $r_2=0.1$, $D_1=0.2$, $D_2=0.1$




F i g. 5. Solutions for two layers: $r_1=r_2=0.1$, $D_1=0.5$, $D_2=0.1$

## Hyperbolic type splines for 1-D nonstationary problem in N layered domains

**Example 5.** We consider the following parabolic type PDEs 1-D initial-value problem in N layers with continuous conditions and BCs in $z$-direction:

$$D_i\frac{\partial^2}{\partial z^2}u_i(z,t)+r_i\frac{\partial}{\partial z}u_i(z,t)=\frac{\partial u_i(z,t)}{\partial t}, \qquad (6)$$

where $z\in[z_{i-1},z_i]$, $t\in[0,t_f]$, $i=\overline{1,N}$, $z_0=0$, $z_N=L$; $r_i\neq 0$ is a constant convection velocity in every layer, $u_i(z,0)=u_{00}$ are the initial conditions. By using transformation $u_i(z,t)=\exp(-r_{i*}(z-z_{i*}))\,v_i(z,t)$ for equation (6) in every layer we obtain following PDEs:

$$\frac{\partial^2}{\partial z^2}v_i(z,t)-r_{i*}^2v_i(z,t)=\frac{\partial v_i(z,t)}{D_i\partial t}, \qquad (7)$$

$z\in[z_{i-1},z_i]$, $t\in[0,t_f]$.

Using BCs (5) and hyperbolic type spline (1) we determine the unknown functions $e_{iz}(t)$ and $m_{iz}(t)$ depending on the mean integral values $v_{iz}(t)$. By using the mean integral values of differential equation (7), we obtain N linear ODEs equations for determination of $v_{iz}(t)$ in the following form:

$0.5\,e_{iz}(t)\,r_{i*}\coth(0.25\,r_{i*}H_i)/H_i-r_{i*}^2v_{iz}(t)=\frac{dv_{iz}(t)}{D_i dt}$, $i=\overline{1,N}$.

**A particular case.** Let us have parameters $N=2$, $L=3$, $H_1=1.8$, $H_2=1.2$, $r_1=0.4$, $r_2=0.1$, $v_{1z}(0)=v_{2z}(0)=0$, $\alpha=100$, $\beta=1000$, $u_0=1$, $u_L=10$, $D_1=0.2$, $D_2=0.1$, $t_f=30$. Then we have the following discontinuous solutions:

$v_{1z}(H_1,t_f)=6.344$, $v_{2z}(H_1,t_f)=33.555$,
$u_1(H_1,t_f)=2.47$, $u_2(H_1,t_f)=9.88$,
$v_1(H_1,t_f)=14.94$, $v_2(H_1,t_f)=24.30$ (see Figs 6 and 7).

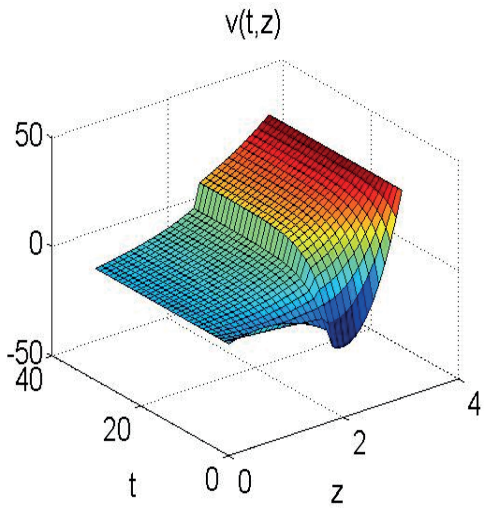



Fig. 6. The nonstationary v-solutions

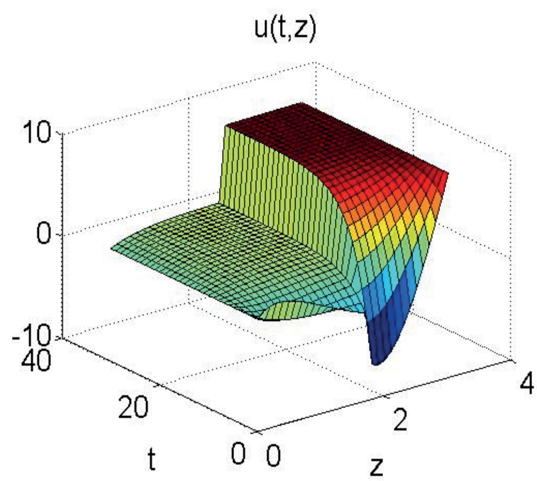



Fig. 7. The nonstationary u-solutions

## Conclusion

The 3-D mass transfer problem in multi-layered domain is reduced to 2-D and 1-D problems using the special integral parabolic, hyperbolic and exponential type splines. These splines are obtained from the general spline with two fixed functions. The parameters of these functions are the characteristic values for the corresponding homogeneous ODEs of second order in fixed direction. These parameters are the best parameters for minimal error. The 1-D differential and discrete problems are solved analytically. For the corresponding diffusion-convection problem the discontinuous solutions are obtained. The solutions for the corresponding averaged non-stationary 3-D initial-boundary value problem are obtained also numerically. The numerical solution is compared with the analytical solution. The maximum absolute value of difference between corresponding numerical and averaged data was in the range of 2–3 percent.

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*Submitted 05.02.2020; revised 27.04.2020; published online 25.05.2020.*

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*Поступила 05.02.2020; принята к публикации 27.04.2020; опубликована онлайн 25.05.2020.*

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