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Original article

Numerical Study of the Effect of Stochastic Disturbances on the Behavior of Solutions of Some Differential Equations

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Abstract

Nowadays interest of the deterministic differential system of Lorentz equations is still primarily due to the problem of gas and fluid turbulence. Despite numerous existing systems for calculating turbulent flows, new modifications of already known models are constantly being investigated.
In this paper we consider the effect of stochastic additive perturbations on the Lorentz convective turbulence model. To implement this and subsequent interpretation of the results obtained, a numerical simulation of the Lorentz system perturbed by adding a stochastic differential to its right side is carried out using the programming capabilities of the MATLAB programming environment.

Keywords: system of Lorentz differential equations, nonlinear dynamics, deterministic chaos, stochastic perturbations.

The authors declare no conflict of interest.

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Оригинальная статья

Численное исследование влияния стохастических возмущений на поведение решений некоторых дифференциальных уравнений

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Аннотация
На сегодняшний день интерес к детерминированной дифференциальной системе уравнений Лоренца по-прежнему обусловлен прежде всего проблемой турбулентности газов и жидкости. Несмотря на большое число существующих систем для расчета турбулентных течений, постоянно исследуются новые модификации уже известных моделей.
В данной работе рассматривается влияние стохастических аддитивных возмущений на модель конвективной турбулентности Лоренца. Для реализации этого и последующей интерпретации полученных результатов, осуществляется численное моделирование системы Лоренца, возмущенной за счет добавления в ее правую часть стохастического дифференциала, с использованием программных возможностей среды программирования MATLAB.

Ключевые слова: система дифференциальных уравнений Лоренца, нелинейная динамика, детерминированный хаос, стохастические возмущения.

Авторы заявляют об отсутствии конфликта интересов.

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Introduction

Nowadays every human faces one of the most difficult challenges, turbulence every day. This thorny issue attracted new scientists year-by-year and as a result of their studies the Lorenz strange attractor was discovered. It was the first example of deterministic chaos.

The Lorenz model [1] was created in 1963 owing to a series of transformations of the Navier–Stokes equations. Its solutions were interesting because of their quasi-stohastic trajectories and absence of external sources of noise. Such solutions for the first time appeared in a deterministic system. Overall the Lorenz model is based on a two-dimensional thermal convection. For the stochastic part of the model, a stochastic differential equation (SDE) will be used. Such differential equations contain a stochastic term, and therefore their solution is also a stochastic process.

This study focuses on modelling and analysis of the stability of the Lorenz system under the influence of stochastic disturbances. In order to realize it and to interpret results, a simulation of the additively disturbed Lorenz system was carried out with MATLAB software package.

Properties of the Lorenz system

Consider the following classical Lorenz equations:

$$\begin{cases} \dot{x} = \sigma\,(y - x) \\ \dot{y} = x\,(r - z) - y \\ \dot{z} = xy - bz \end{cases} \qquad (1)$$

The variable x represents the rotation rate of the Rayleigh-Benard convection cells, y characterizes the temperature difference ΔT between rising and descending fluid, z shows the deviation of the vertical temperature profile from the linear relationship. The parameters σ, r, b reflect the values of the Prandtl number, the Rayleigh number, and the coefficient linked to the geometry of the area respectively.

As well known the Lorenz system has the following properties:

1. Homogeneity. The first and most obvious property.
2. Symmetry. In the phase space symmetry is obvious after $x \to (-x),\, y \to (-y)$.
3. Dissipation. In three-dimensional phase space (x, y, z) we will consider vector of speeds $\underline{\vec{L}}(x'_t, y'_t, z'_t)$. Its negative divergence characterizes dissipative system

$$div\vec{L} = \frac{\partial}{\partial x}(\sigma y - \sigma x) + \frac{\partial}{\partial y}(rx - y - x) + \frac{\partial}{\partial z}(xy - bz) = -\sigma - 1 - b < 0 \quad (2)$$

Let's look at set of Lorenz systems with different initial conditions. They take volume ΔV while $t = 0$. During the evolution of the system volume declines according to $\Delta V = V_0 \exp(-\sigma - b - 1)$. At $t \to \infty$ all phase-space trajectories are concentrated inside a compact attractor.

Then we will check the Lorenz system for fixed points:

$$\begin{cases} \sigma(y - x) = 0 \\ x(r - z) - y = 0 \\ xy - bz = 0 \end{cases} \Leftrightarrow \begin{cases} x = y \\ x(r - 1 - z) = 0 \\ x^2 = bz \end{cases} \Leftrightarrow \begin{cases} x = y \\ \left[\begin{array}{l} x = 0 \\ z = r - 1 \end{array}\right. \\ x^2 = bz \end{cases} \qquad (3)$$

The Lorenz system always has fixed point $P_0(0,0,0)$. Also when $r > 1$ two other fixed points appear $P_1(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $P_2(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$.

Point $r = 1$ is a bifurcation point. At $r < r_1 \approx 13{,}926$ separatrices S_1 and S_2 attract to the nearest fixed points P_1 and P_2. At $r = r_1$ separatrices transform into a homoclinic loops. They afterwards transform into the saddle orbits, borders of attraction area of P and P_2. Also separatrices S_1 and S_2 approaches to P_2 and P_1 accordingly. More detailed information about the structure of the Lorenz system can be found in various monographs [2-9]. The most interesting situation appears at $r = r_2 \simeq 24{,}06$. It corresponds to well-known Lorenz strange attractor, which has property of strong dependence on initial conditions. It means that any small change in the coordinates of the initial point leads to completely different solution.

Ito's stochastic calculus

We will describe stochastic differential equations (SDE) with Ito's stochastic calculus. It is based on a stochastic Wiener process. Overall, stochastic process is a set of random variables that has been indexed by some parameter such as time.

Initially we consider division $\{\tau_j^{(N)}\}$ of a $[0, T]$, which corresponds to $0 = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T$ with $\Delta = \max\limits_{0<j<N-1} \left|\tau_{j+1}^{(N)} - \tau_j^{(N)}\right| \to 0$.

Then we determine sequence of functions in the following way: $\xi^{(N)}(t,\omega) = \xi(\tau_j^{(N)}, \omega)$ at $t \in [\tau_j^{(N)}, \tau_{j+1}^{(N)})$, $j = 0, 1, \ldots, N-1$.

Definition: Stochastic Ito integral for ξ_t is a convergence in quadratic mean of following expression, where f_τ is a Wiener process [10-12]:

$$\lim_{N\to\infty} \sum_{j=0}^{N-1} \xi^{(N)}(t,\omega)\left(f(\tau_{j+1}^{(N)}, \omega) - f(\tau_j^{(N)}, \omega)\right) \underset{def}{=} \int_0^T \xi_\tau \, df_\tau \,. \qquad (4)$$

As a result we need to determine multiple stochastic integrals for introduction of a numerical scheme. Let's determine them by the following expression:

$$I^{(i_1 \ldots i_k)}_{l_1 \ldots l_{k_{s,t}}} = \begin{cases} \int_\tau^s (s - \tau_k)^{l_k} \ldots \int_\tau^{\tau_2} (s - \tau_1)^{l_1} \, df^{(i_1)}_{\tau_1} \ldots df^{(i_k)}_{\tau_k}, \text{ at } k > 0, \\ 1, \text{ at } k = 0. \end{cases} \qquad (5)$$

The simulated Lorenz system is demonstrated below:

$$d\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} \sigma(y - x) \\ x(r - z) - y \\ xy - bz \end{bmatrix} dt + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix} d\begin{bmatrix} W_1^{(t)} \\ W_2^{(t)} \\ W_3^{(t)} \end{bmatrix}. \qquad (6)$$

In this paper we used the version of unified Taylor-Ito expansion gained by Kulchitskiy[1] [13]. The main problem is that this expansion contains multiple stochastic integrals, which are not easily approximated [14, 15]. We will use the fundamental results of

[1] Kulchitskiy O.Yu., Kuznetsov D.F. Numerical simulation of stochastic systems of linear stationary differential equations. *Journal Differential Equations and Control Processes*. 1998; (1):41-65. Available at: https://www.elibrary.ru/item.asp?id=25301726 (accessed 04.02.2021). (In Russ.)

Kuznetsov [16, 17] to approximate these integrals properly[2]. He discovered expansions of our multiple stochastic integrals using independent random variables ξ_j.
We will use several of them:

$$I_{0_{T,t}}^{(i_1)} = \sqrt{T-t}\,\xi_0, \tag{7}$$

$$I_{1_{T,t}}^{(i_1)} = -\frac{(T-t)^{3/2}}{2}\left(\xi_0 + \frac{1}{\sqrt{3}}\xi_1\right), \tag{8}$$

$$I_{2_{T,t}}^{(i_1)} = \frac{(T-t)^{5/2}}{2}\left(\xi_0 + \frac{\sqrt{3}}{2}\xi_1 + \frac{1}{2\sqrt{5}}\xi_2\right). \tag{9}$$

Using them in the Taylor-Ito expansion in the Kloeden-Platen form [18, 19], we get the explicit numerical scheme directly from this expansion. For the sake of brevity we only present here the final result. Initially let us denote step of division $\{\tau_j\}_{j=0}^{N}$ as h, $j=1...N$.
The explicit numerical scheme, which we have implemented, is as follows:

$$x_{j+1} = x_j + he + \frac{h^2}{2}(-he+\sigma g) + \frac{h^3}{6}e_1 - h^{5/2}\sigma x_j c v_1, \tag{10}$$

$$y_{j+1} = y_j + hg + \frac{h^2}{2}\left((r-z_j)e - g - x_j f\right) - \frac{h^3}{6}g_1 - \\ - h^{3/2}cx_jv_2 + h^{5/2}\left(-e+(1+b)x_j\right)cv_1 + h^{5/2}ecv_3, \tag{11}$$

$$z_{j+1} = z_j + hf + \frac{h^2}{2}ey_j + gx_j - bf + \frac{h^3}{6}f_1 + \\ + h^{1/2}c\xi_1 - h^{3/2}bcv_2 + h^{5/2}c(b^2-x_j-2)v_1. \tag{12}$$

In the scheme (10)-(12) we made a number of some designations to simplify the recording of the scheme that was written above:
$e = -\sigma x_j + \sigma y_j$, $g = rx_j - y_j - x_jz_j$, $f = -bz_j + x_jy_j$,

$$g_1 = e\left((-\sigma-1)(r-z_j)\right) + bz_j - 2x_jy_j + g\left(\sigma(r-z_j)+1-x_j^2\right) + f\left(-e+(b+1)x_j\right),$$

$$f_1 = e\left(2x_j(r-z_j) - (b+1+\sigma)y_j\right) + g\left(-(b+1+\sigma)x_j + 2\sigma y_j\right) + f\left(b^2 - x_j^2\right),$$

$$v_1 = \frac{\xi_1}{6} - \frac{\xi_2}{4\sqrt{3}} + \frac{\xi_3}{6\sqrt{20}},\quad v_2 = -\frac{\xi_1}{2} - \frac{\xi_2}{2\sqrt{3}},\quad v_3 = -\frac{\xi_1}{6} - \frac{\xi_3}{3\sqrt{20}}.$$

Results of numerical modeling

It was decided to start with intermediate values to understand how the system as a whole would behave. First the parameter $r=20$ was fixed and two situations were modelled: at $c=2$ and at $c=0$. Parameter c shows the intensity of stochastic influence. The state at $c=0$ is given for comparison (Figure 1).

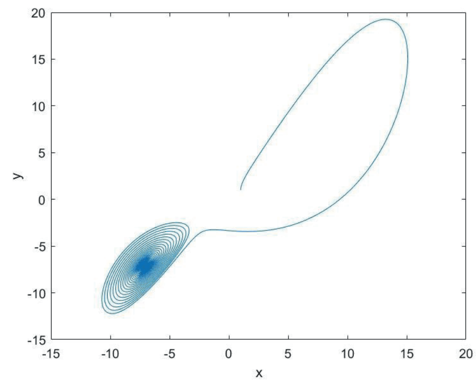

Fig. 1. $r=20, c=0$

At $c=2$ the trajectory loses its regularity, which is reasonably predictable. Further, let us increase c to 3.

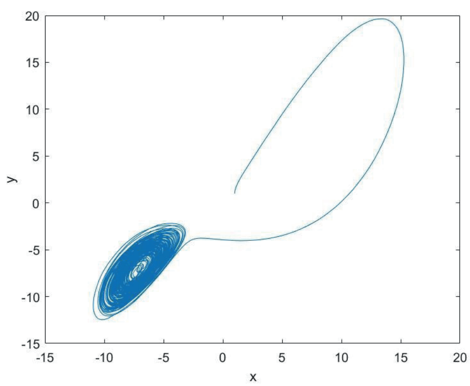

Fig. 2. $r=20, c=2$

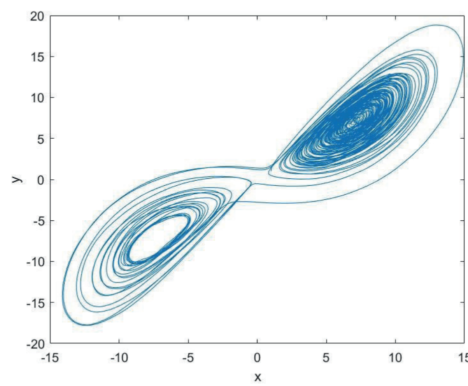

Fig. 3. $r=20, c=3$

At $c=2$ the trajectory loses its regularity, which is quite predictable (Figure 2). Further, let us increase c to 3 (Figure 3). It turns out that the trajectory of the interfered system seems like the Lorenz attractor, while r value is sufficiently far from 24.06. Next, let us increase the parameter c to 4, to test this assumption, and get a picture that is even more similar to Lorenz attractor (Figure 4).

[2] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. 3-rd ed. SPbPU Publ., SPb; 2009. (In Russ.)

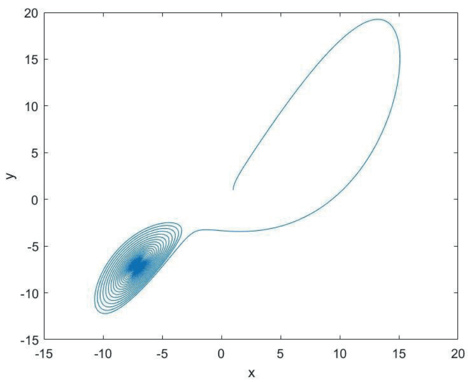

Fig. 4. $r = 20, c = 4$

Then consider a different state of the system at $r = 13$ and look at the effect of noise, but in three-dimensional space.

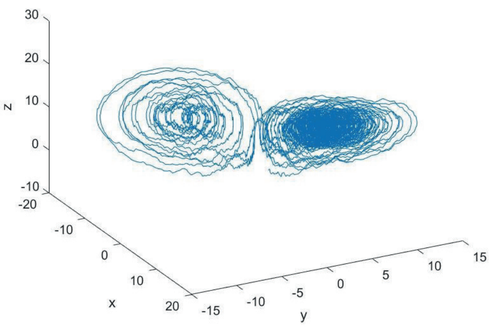

Fig. 5. $r = 13, c = 4$

As be seen from the graph, with less r perturbed systems also demonstrates similar behavior. Under these conditions, the change of attractor occurs much earlier than in a classic system. As stochastic intensity increases, the stochastic analogue of the Lorenz attractor with substantially smaller r can be observed. Overall there is an negative relationship between the stochastic factor and the bifurcation values of r. It is interesting to see how the system works with large values of r. We start with $r = 200$ and build a determine system (blue color) and interfered system with $c = 5$.

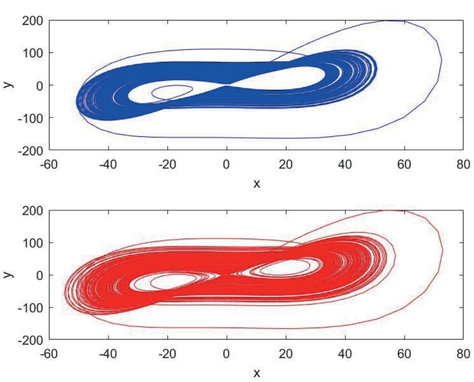

Fig. 6. $r = 200, c = 0, c = 5$

The graphs are quite similar, and here we clearly see auto-oscillating mode. By increasing r to 300 (Figure 7), and then up to 500 (Figure 8), we can obtain a predictable result, based on fact that r is an analogue of the Rayleigh number.

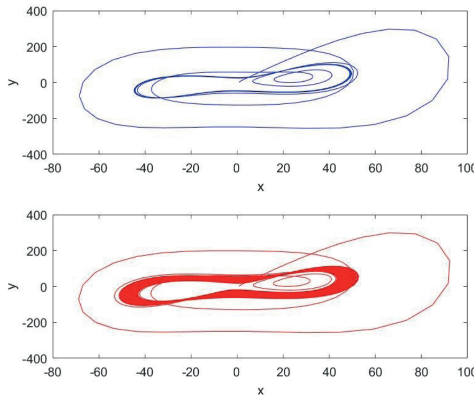

Fig. 7. $r = 200, c = 0, c = 5$

As r increases, the role of noise will gradually decrease. The system will be a stochastic analogue of the auto-oscillating movement, which will differ from the calm system only by a slight irregularity of the trajectory.

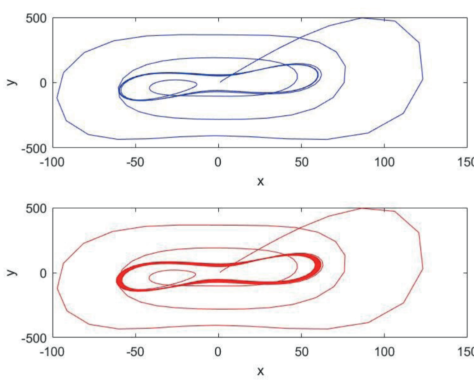

Fig. 8. $r = 200, c = 0, c = 5$

Conclusions

In conclusion we would like to make the following observations and draw a parallel with the real physical system. All in all, it seems quite logical that stochastic interferences strengthen quasi-stochastic oscillations around equilibrium positions. As a result a trajectory similar enough to the Lorenz strange attractor appears at smaller r. The same changes can be observed, for example, in real physical systems, where turbulence occurs earlier in the presence of some noise source than without it. Then, gradually, the noise reduces effect on the system, because the Rayleigh number is already high enough.

The behavior of the system after the noise appearance demonstrates quite clearly that stochastic interference plays a significant role in describing turbulence. Lorenz [1] wanted to use his model for long-term weather forecasting. Moreover, he wanted to prove the theoretical existence of such a method. By and large, due to the significant impact of additive interference, it is unlikely that such a method will ever be developed.

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