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Original article

An Approximate Method for Solving Boundary Value Problems with Moving Boundaries in the Developed Software Package TB-Analisys

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Abstract

The problem of vibrations of objects with moving boundaries, formulated as a differential equation with boundary and initial conditions, is a nonclassical generalization of a hyperbolic problem. To facilitate the construction of the solution to this problem and to justify the choice of the form of the solution, equivalent integro-differential equations with symmetric and time-dependent kernels and time-varying integration limits are constructed. The advantages of the method of integro-differential equations are revealed in the transition to more complex dynamic systems carrying concentrated masses vibrating under the action of moving loads. The method is extended to a wider class of model boundary value problems that take into account the bending stiffness, the resistance of the external environment, and the stiffness of the base of an oscillating object. Particular attention is paid to the consideration of the most common case in practice, when external disturbances act at the borders. The solution is made in dimensionless variables accurate to second-order values of relatively small parameters characterizing the velocity of the boundary. An approximate solution is found to the problem of transverse vibrations of a rope of a lifting device, which has bending rigidity, one end of which is wound on a drum, and a load is fixed on the other. The results obtained for the amplitude of oscillations corresponding to the nth dynamic mode are presented. The phenomenon of steady-state resonance and passage through resonance is investigated using numerical methods. The solution is made in dimensionless variables using the TB-Analisys software package developed in the Matlab environment, which allows using the results obtained for calculating a wide range of technical objects.

Keywords: resonance properties, oscillations of systems with moving boundaries, laws of motion of boundaries, integro-differential equations, amplitude of oscillations

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Оригинальная статья

Приближенный метод решения краевых задач с подвижными границами в разработанном программном комплексе TB-Analisys

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Аннотация

Задача о колебаниях объектов с движущимися границами, сформулированная в виде дифференциального уравнения с граничными и начальными условиями, является неклассическим обобщением гиперболической задачи. Для облегчения построения решения этой задачи и обоснования выбора формы решения построены эквивалентные интегро-дифференциальные уравнения с симметричными и зависящими от времени ядрами и изменяющимися во времени пределами интегрирования. Преимущества метода интегро-дифференциальных уравнений раскрываются при переходе к более сложным динамическим системам, несущим сосредоточенные массы, колеблющиеся под действием подвижных нагрузок. Метод распространяется на более широкий класс модельных краевых задач, которые учитывают жесткость при изгибе, сопротивление внешней среды и жесткость основания колеблющегося объекта. Особое внимание уделяется рассмотрению наиболее распространенного на практике случая, когда на границах действуют внешние возмущения. Решение принимается в безразмерных переменных с точностью до значений второго порядка относительно малых параметров, характеризующих скорость границы. Найдено приближенное решение задачи о поперечных колебаниях каната подъемного устройства, обладающего жесткостью на изгиб, один конец которого намотан на барабан, а на другом закреплен груз. Представлены результаты, полученные для амплитуды колебаний, соответствующей n-му динамическому режиму. Явление установившегося резонанса и прохождения через резонанс исследуется с помощью численных методов. Решение выполнено в безразмерных переменных с использованием программного пакета TB-Analysis, разработанного в среде Matlab, что позволяет использовать полученные результаты для расчета широкого спектра технических объектов.

Ключевые слова: резонансные свойства, колебания систем с подвижными границами, законы движения границ, интегро-дифференциальные уравнения, амплитуда колебаний

Авторы заявляют об отсутствии конфликта интересов.

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1. Introduction

Among all the many problems of the dynamics of elastic systems from the point of view of technical applications, the problems of oscillations in systems with time-varying geometric dimensions are very relevant. Systems, the boundaries of which are moving, are widespread in technology (ropes of hoisting installations [1-3, 7, 8, 11, 13, 18, 23, 24], flexible transmission links [7, 11, 12], solid fuel rods [9, 10], drill strings [9], etc.). The studies of many authors on the dynamics of lifting ropes have led to the need to formulate new problems in mechanics concerning the dynamics of one-dimensional objects of variable lengths. In the mathematical formulation, this is reduced to new problems of mathematical physics – to the study of the corresponding equations of hyperbolic type in variable ranges of variation of both arguments. The presence of moving boundaries causes significant difficulties in the description of such systems; therefore, approximate solution methods are mainly used here [1-5, 7-19].

Of the analytical methods, the most effective is the method proposed in [6, 20], which consists in the selection of new variables that stop the boundaries and leave the wave equation invariant. In [21], the solution is sought in the form of a superposition of two waves traveling towards each other. The method used in [22], which consists in replacing the geometric variable with a purely imaginary variable, is also effective, which makes it possible to reduce the wave equation to the Laplace equation and apply the methodology of the theory of functions of a complex variable for the solution. However, the exact solution methods are limited by the wave equation and relatively simple boundary conditions.

Of the approximate methods, the most effective method is the Kantorovich-Galerkin method [9, 14, 25], as well as the method for constructing solutions of integro-differential equations described in this paper. The problem of vibrations of objects with moving boundaries, formulated as a differential equation with boundary and initial conditions, is a nonclassical generalization of a hyperbolic problem. To facilitate the construction of the solution to this problem and to justify the choice of the form of the solution, equivalent integro-differential equations with symmetric and time-dependent kernels and time-varying integration limits are constructed. The construction of integro-differential equations of motion of objects of variable length is based on the direct integration of differential equations in combination with the standard replacement of the required function with a new variable.

In trivial cases, the methods of integral equations do not have an advantage over the method of differential equations when applied to the study of oscillations of a system with an infinite number of degrees of freedom [7, 11]. The advantages of the method of integro-differential equations are revealed in the transition to more complex dynamic systems carrying concentrated masses, vibrating under the action of moving loads, etc. These methods can be very fruitful when applied to the dynamics of strands of variable lengths and other mechanical objects with varying boundaries.

In this paper, the method of constructing solutions of integro-differential equations is extended to a wider class of model boundary value problems that take into account the flexural rigidity of an oscillating object [8, 10, 13], the resistance of the external environment [9] and the rigidity of the base (substrate) of the object [7, 11] ... Particular attention is paid to the consideration of the most common case in practice, when external disturbances act at the borders. For a fixed length of the object, the constructed integro-differential equations go over into the classical Fredholm equations of the second kind. The solution is made in dimensionless variables using the TB-Analisys[1] software package developed in the Matlab environment, which allows using the results obtained for calculating a wide range of technical objects.

2. Formulation of the Problem

The differential equation of motion of mechanical objects of variable length has the form

$$U_{\tau\tau}(\xi,\tau)+L[U(\xi,\tau)]=\varphi(\xi,\tau). \qquad (1)$$

We write the boundary conditions in the following form

$$Y_{ji}\left[U\left(\ell_j(\varepsilon\tau),\tau\right)\right]=F_{ji}(\tau); \qquad (2)$$

$$i=\overline{1,m};\ j=\overline{1,2}.$$

Here $U(\xi,\tau)$ – offset function; L – linear homogeneous differential operator with respect to ξ order $2m$ ($m'' 2$ – positive integer); Y_{ji} — linear homogeneous differential operators with respect to ξ; $\varphi(\xi,\tau)$, $F_{ji}(\tau)$ – specified class functions C and C^2 respectively; $\ell_j(\varepsilon\tau)$ – boundary motion law; ε – small parameter ($\varepsilon=V/a$, V – border speed, a – vibration propagation speed).

The movement of the boundaries according to the law $\ell_j(\varepsilon\tau)$ corresponds to the slow movement mode.

When analyzing the resonance properties, the initial conditions are taken in the form $U(\xi,0)=U_\tau(\xi,0)=0$.

To eliminate in homogeneities in the boundary conditions, a new function is introduced into equation (1)

$$U(\xi,\tau)=V(\xi,\tau)+H(\xi,\tau), \qquad (3)$$

where

$$H(\xi,\tau)=\sum_{k=1}^{2}\sum_{r=1}^{m}D_{kr}(\xi,\varepsilon\tau)F_{kr}(\tau), \qquad (4)$$

the function satisfies the equation

$$L[D_{kr}(\xi,\varepsilon\tau)]=0 \qquad (5)$$

and conditions

$$Y_{ji}\left[D_{kr}\left(\ell_j(\varepsilon\tau),\tau\right)\right]=\begin{cases}1, & k=j\wedge r=i;\\ 0, & k\neq j\vee r\neq i.\end{cases}$$

Substituting (3) into equation (1) taking into account (4), (5), the function $V(\xi,\tau)$ is found as a solution to the following problem:

$$V_{\tau\tau}(\xi,\tau)+L\left[V(\xi,\tau)\right]=\varphi(\xi,\tau)-H_{\tau\tau}(\xi,\tau); \qquad (6)$$

$$Y_{ji}\left[V\left(\ell_j(\varepsilon\tau),\tau\right)\right]=0. \qquad (7)$$

In [11], an integro–differential equation corresponding to problem (6), (7) was obtained in the form

$$V(\xi,\tau)=-\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}K(\xi,\zeta,\varepsilon\tau)\left[V_{\tau\tau}(\zeta,\tau)-\varphi(\zeta,\tau)+H_{\tau\tau}(\zeta,\tau)\right]d\zeta, \qquad (8)$$

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where $K(\xi,\zeta,\varepsilon\tau)$ – symmetrical in ξ and ζ kernel time–dependent via parameter $\varepsilon\tau$.

Theorem 1. *In a time interval $\Delta\tau$, commensurate with unity, the equation of oscillations of an object with a fixed parameter $l=l(\tau_0)=const$ differs from the corresponding equation of oscillations of an object with a variable parameter $l=l(\tau)$ by terms proportional to the factor ε , provided that the derivative of the kernel $K(x,s,l)$ with respect to the parameter $l(\tau)$ is bounded.*

Proof. Let us expand the right-hand side of Eq. (8) in terms of the parameter $l(\tau)$ in the vicinity of some fixed value of the dimensionless length $l(\tau_0)$ in a Taylor series.

Assuming

$$l(\tau_0+\Delta\tau)=l(\tau_0)+\Delta l(\tau)+\ldots$$

get

$$\begin{aligned} V(\xi,\tau)=&-\int_0^{l(\tau_0)}K(\xi,\zeta,l(\tau_0))\left[V_{\tau\tau}(\zeta,\tau)-\varphi(\zeta,\tau)\right]d\zeta-\\ &-\Delta l(\tau)\left\{K(\xi,l(\tau_0),l(\tau_0))\left[V_{\tau\tau}(l(\tau_0),\tau)-\varphi(l(\tau_0),\tau)\right]+\right.\\ &\left.+\int_0^{l(\tau_0)}\frac{\partial K(\xi,\zeta,l(\tau_0))}{\partial l(\tau)}\left[V_{\tau\tau}(\zeta,\tau)-\varphi(\zeta,\tau)\right]d\zeta\right\}-\\ &-\frac{(\Delta l(\tau))^2}{2!}\left[\frac{\partial K(\xi,l(\tau_0),l(\tau_0))}{\partial l(\tau)}\ldots\right]. \end{aligned} \tag{9}$$

We will assume that the function $l(\tau)$ is a function of slow time $l=l(\tau_1)$, $\tau_1=\varepsilon\tau$, i.e. is a function of time, the time derivative of which is proportional to some small parameter ε . The differential of the length of the object $\Delta l(\tau_1)$ in accordance with the rule of differentiation of the slow time function [7, 11] is calculated by the formula $\Delta l(\tau_1)=\varepsilon\dfrac{d\,l(\tau_1)}{d\tau_1}\Delta\tau.$

Let us choose the time interval $\Delta\tau$ in the form

$$\Delta\tau=\theta(\tau), \tag{10}$$

where $\theta(\tau)-$ is some function of order unity.

Substituting (10) into (9), we find that in a time interval $\Delta\tau$, of the order of unity, expansion (9) has the form

$$\begin{aligned} V(\xi,\tau)=&-\int_0^{l(\tau_0)}K(\xi,\zeta,l(\tau_0))\left[V_{\tau\tau}(\zeta,\tau)-\varphi(\zeta,\tau)\right]d\zeta-\\ &-\varepsilon l'(\tau)\theta(\tau)\left\{K(\xi,l(\tau_0),l(\tau_0))\left[V_{\tau\tau}(l(\tau_0),\tau)-\varphi(l(\tau_0),\tau)\right]+\right.\\ &\left.+\int_0^{l(\tau_0)}\frac{\partial K(\xi,\zeta,l(\tau_0))}{\partial l(\tau)}\left[V_{\tau\tau}(\zeta,\tau)-\varphi(\zeta,\tau)\right]d\zeta\right\}-\\ &-\varepsilon^2 l'^2(\tau)\frac{\theta(\tau)}{2!}\left[\frac{\partial K(\xi,l(\tau_0),l(\tau_0))}{\partial l(\tau)}\ldots\right]. \end{aligned}$$

Taking into account the condition of the theorem on the boundedness of the derivative of the kernel $K(x,s,l)$ with respect to the parameter $l(\tau)$ and comparing the results obtained, we find that an equation with a fixed parameter $l=l(\tau_0)=const$ differs from an equation with a variable parameter in the range of $\Delta\tau\sim 1$ terms proportional to the factor. This proves the theorem.

3. The Solution of the Problem

The solution to problem (9) will be sought in the form of a series:

$$V(\xi,\tau)=\sum_{n=1}^{\infty}f_n(\tau)X_n(\xi,\varepsilon\tau), \tag{11}$$

where $X_n(\xi,\varepsilon\tau)$ – eigenfunctions, which are formally constructed solutions of the integral equation

$$X_n(\xi,\varepsilon\tau)=\omega_{0n}^2(\varepsilon\tau)\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}K(\xi,\zeta,\varepsilon\tau)X_n(\zeta,\varepsilon\tau)d\zeta, \tag{12}$$

where $\varepsilon\tau$ considered as a parameter.

Solution (11) is exact if the boundaries are motionless.

Own functions $X_n(\xi,\varepsilon\tau)$ satisfy the boundary conditions (8) and in this case play the role of dynamic modes.

Using the results of [11], we expand the symmetric in ξ and ζ kernel in a row by its own functions $X_n(\xi,\varepsilon\tau)$:

$$K(\xi,\zeta,\varepsilon\tau)=\sum_{n=1}^{\infty}\frac{X_n(\xi,\varepsilon\tau)X_n(\zeta,\varepsilon\tau)}{\omega_{0n}^2(\varepsilon\tau)}, \tag{13}$$

where $\omega_{0n}(\varepsilon\tau)-$ natural frequencies of the problem determined by the formula

$$\frac{1}{\omega_{0n}^2(\varepsilon\tau)}=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}K(\xi,\zeta,\varepsilon\tau)X_n(\xi,\varepsilon\tau)X_n(\zeta,\varepsilon\tau)d\xi d\zeta. \tag{14}$$

Let us differentiate series (11) by time:

$$V_\tau(\xi,\tau)=\sum_{n=1}^{\infty}\left[f_n'(\tau)X_n(\xi,\varepsilon\tau)+\varepsilon X_{n_\tau}(\xi,\varepsilon\tau)f_n(\tau)\right].$$

After repeated differentiation, we get

$$V_{\tau\tau}(\xi,\tau)=\sum_{n=1}^{\infty}\left\{f_n''(\tau)X_n(\xi,\varepsilon\tau)+2\varepsilon X_{n_\tau}(\xi,\varepsilon\tau)f_n'(\tau)+\varepsilon^2 X_{n_{\tau\tau}}(\xi,\varepsilon\tau)f_n(\tau)\right\}. \tag{15}$$

Substitute series (11), (13), (15) into equation (9) taking into account the orthogonality of the functions $X_n(\xi,\varepsilon\tau)$ on the interval $\left[\ell_1(\varepsilon\tau);\ell_2(\varepsilon\tau)\right]$ with weight $g(\xi)$ and replacements

$$f_n(\tau)=\mu_n(\tau)+\sum_{k=1}^{2}\sum_{r=1}^{m}Q_{n_{kr}}(\varepsilon\tau)F_{kr}(\tau),$$

where

$$Q_{n_{kr}}(\varepsilon\tau)=-\frac{\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}D_{kr}(\xi,\varepsilon\tau)X_n(\xi,\varepsilon\tau)g(\xi)d\xi}{\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}X_n^2(\xi,\varepsilon\tau)g(\xi)d\xi}.$$

Note that if we expand the function $H(\xi,\tau)$ in a Fourier series:

$$H(\xi,\tau)=\sum_{n=1}^{\infty}\varphi_n(\tau)X_n(\xi,\varepsilon\tau), \tag{16}$$

where

$$\varphi_n(\tau)=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}H(\xi,\tau)X_n(\xi,\varepsilon\tau)(\xi)d\xi\Big/\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}X_n^2(\xi,\varepsilon\tau)g(\xi)d\xi,$$

here $g(\xi)-$ is a weight function, then the replacement can be made in a simpler form

$$f_n(\tau)=\mu_n(\tau)-\varphi_n(\tau).$$

In resonance phenomena, the amplitudes of all dynamic modes, except for the resonant one, are small. Therefore, the nonresonant terms of series (11), (15) are neglected due to their smallness. In this case, we obtain a split system of ordinary differential equations with variable coefficients

$$A_{1n}(\varepsilon\tau)\mu_n''(\tau)+2\varepsilon A_{2n}(\varepsilon\tau)\mu_n'(\tau)+\varepsilon^2 A_{3n}(\varepsilon\tau)\mu_n(\tau)+A_{1n}(\varepsilon\tau)\omega_{0n}^2(\varepsilon\tau)\mu_n(\tau)=\theta_n(\tau), \quad (17)$$

where

$$\theta_n(\tau)=E_n(\tau)-2\varepsilon\sum_{k=1}^{2}\sum_{r=1}^{m}B_{n_{kr}}(\varepsilon\tau)F_{kr}'(\tau)-\varepsilon^2\sum_{k=1}^{2}\sum_{r=1}^{m}C_{n_{kr}}(\varepsilon\tau)\,F_{kr}(\varepsilon\tau)- \quad (18)$$

$$-\omega_{0n}^2(\varepsilon\tau)A_{1n}(\varepsilon\tau)\sum_{k=1}^{2}\sum_{r=1}^{m}Q_{n_{kr}}(\varepsilon\tau)F_{kr}(\varepsilon\tau).$$

Here:

$$A_{1n}(\varepsilon\tau)=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}X_n^2(\xi,\varepsilon\tau)g(\xi)d\xi; \quad (19)$$

$$\varepsilon A_{2n}(\varepsilon\tau)=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}X_{n_\tau}(\xi,\varepsilon\tau)X_n(\xi,\varepsilon\tau)g(\xi)d\xi; \quad (20)$$

$$\varepsilon^2 A_{3n}(\varepsilon\tau)=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}X_{n_{\tau\tau}}(\xi,\varepsilon\tau)X_n(\xi,\varepsilon\tau)g(\xi)d\xi; \quad (21)$$

$$\varepsilon B_{n_{kr}}(\varepsilon\tau)=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}[Q_{n_{kr}}(\varepsilon\tau)X_n(\xi,\varepsilon\tau)+D_{kr}(\xi,\varepsilon\tau)]_\tau\cdot X_n(\xi,\varepsilon\tau)g(\xi)d\xi; \quad (22)$$

$$\varepsilon^2 C_{n_{kr}}(\varepsilon\tau)=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}[Q_{n_{kr}}(\varepsilon\tau)X_n(\xi,\varepsilon\tau)+D_{kr}(\xi,\varepsilon\tau)]_{\tau\tau}\cdot X_n(\xi,\varepsilon\tau)g(\xi)d\xi;$$

$$E_n(\tau)=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}\varphi(\xi,\tau)X_n(\xi,\varepsilon\tau)g(\xi)d\xi.$$

System (17) up to values of the order of smallness ε^2 will look like

$$A_{1n}(\varepsilon\tau)\mu_n''(\tau)+2\varepsilon A_{2n}(\varepsilon\tau)\mu_n'(\tau)+A_{1n}(\varepsilon\tau)\omega_{0n}^2(\varepsilon\tau)\mu_n(\tau)=\theta_n(\tau), \quad (23)$$

where

$$\theta_n(\tau)=\omega_{0n}^2(\varepsilon\tau)A_{1n}(\varepsilon\tau)\varphi_n(\tau)+E_n(\tau).$$

Taking into account (11), (16) solution (3) will have the form:

$$U(\xi,\tau)=\sum_{n=1}^{\infty}\mu_n(\tau)X(\xi,\varepsilon\tau)+\sum_{k=1}^{2}\sum_{r=1}^{m}F_{kr}(\tau)[D_{kr}(\xi,\varepsilon\tau)+\sum_{n=1}^{\infty}Q_{n_{kr}}(\varepsilon\tau)X_n(\xi,\varepsilon\tau)]. \quad (24)$$

Theorem 2. The solution to problem (1) - (2) can be represented in the form

$$U(\xi,\tau)=\sum_{n=1}^{\infty}\mu_n(\tau)X_n(\xi,\varepsilon\tau).$$

Proof. The quantities $Q_{n_{kr}}(\varepsilon\tau)$ determined by the expression

$$Q_{n_{kr}}(\varepsilon\tau)=-\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}D_{kr}(\xi,\varepsilon\tau)X_n(\xi,\varepsilon\tau)g(\xi)d\xi\Big/\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}X_n^2(\xi,\varepsilon\tau)g(\xi)d\xi,$$

are for the function $-D_{kr}(\xi,\varepsilon\tau)$ the coefficients of the Fourier series expansion in the system of orthogonal weights $g(\xi)$ eigenfunctions $X_n(\xi,\varepsilon\tau)$ on the interval $[\ell_1(\varepsilon\tau),\ \ell_2(\varepsilon\tau)]$, i.e.

$$\sum_{n=1}^{\infty}Q_{n_{kr}}(\varepsilon\tau)X_n(\xi,\varepsilon\tau)=-D_{kr}(\xi,\varepsilon\tau).$$

Therefore, the expression in square brackets of equality (24) is equal to zero. The theorem is proved.

For simplicity, we introduce into equation (23) a new function

$$\mu_n(\tau)=A_{0n}(\varepsilon\tau)y_n(\tau), \quad (25)$$

where

$$A_{0n}(\varepsilon\tau)=\exp\left[-\int_0^\tau\frac{\varepsilon A_{2n}(\varepsilon\tau)}{A_{1n}(\varepsilon\tau)}d\tau\right]. \quad (26)$$

Then equation (23) will not contain a term with $y'(\tau)$

$$y_n''(\tau)+\omega_{0n}^2(\varepsilon\tau)y_n(\tau)=\theta_n(\tau)/[A_{0n}(\varepsilon\tau)A_{1n}(\varepsilon\tau)] \quad (27)$$

Let be

$$\varphi(\xi,\tau)=B_0(\xi)\cos W_0(\tau); \quad (28)$$

$$F_{ji}(\tau)=B_{ji}\cos W_{ji}(\tau);\ \ j=\overline{1,2};\ \ i=\overline{1,m}, \quad (29)$$

where B_{ji} constant values; $W_0(\tau), W_{ji}(\tau)$ monotonically increasing functions; $B_0(\xi)$ — function characterizing the intensity of the distributed load.

Taking into account equalities (28), (29), expression (18) takes the form

$$\frac{\theta_n(\tau)}{A_{0n}(\varepsilon\tau)A_{1n}(\varepsilon\tau)}=M_{n0}(\varepsilon\tau)\cos W_0(\tau)+\sum_{j=1}^{2}\sum_{i=1}^{m}M_{nji}(\varepsilon\tau)\cos W_{nji}(\tau), \quad (30)$$

where

$$M_{n0}(\varepsilon\tau)=\int_{\ell_1(\varepsilon\tau)}^{\ell_2(\varepsilon\tau)}B_0(\xi)X_n(\xi,\varepsilon\tau)g(\xi)d\xi\,/\,[A_{0n}(\varepsilon\tau)A_{1n}(\varepsilon\tau)];$$

$$M_{nji}(\varepsilon\tau)=\frac{-B_{ji}\omega_{0n}^2(\varepsilon\tau)Q_{nji}(\varepsilon\tau)}{A_{0n}(\varepsilon\tau)}. \quad (31)$$

We restrict ourselves to considering the case when expression (30) has the form

$$\theta_n(\tau)/\left[A_{0n}(\varepsilon\tau)A_{1n}(\varepsilon\tau)\right]=M_n(\varepsilon\tau)\cos W_n(\tau), \quad (32)$$

where $W_n(\tau)$ – monotonically increasing function.

Equality (30) can be replaced by equality (32) in the cases described in [8, 9, 14].

Taking into account the above, equation (27) takes the form

$$y_n''(\tau)+\omega_{0n}^2(\varepsilon\tau)y_n(\tau)=M_n(\varepsilon\tau)\cos W_n(\tau). \quad (33)$$

Solution of equation (33) with zero initial conditions $y(0)=0;\ y'(0)=0$ has the form [12]:

$$y_n(\tau)=\int_0^\tau\gamma_n(\tau,\zeta)M_n(\varepsilon\zeta)\cos W_n(\zeta)d\zeta, \quad (34)$$

where

$$\gamma_n(\tau,\zeta)=\frac{-y_{1n}(\tau)y_{2n}(\zeta)-y_{1n}(\zeta)y_{2n}(\tau)}{y_{1n}(\zeta)y_{2n}'(\zeta)-y_{1n}'(\zeta)y_{2n}(\zeta)},$$

y_{1n}, y_{2n} — linearly independent solutions of the homogeneous equation corresponding to (33), i.e.

$$y_n''(\tau)+\omega_{0n}^2(\varepsilon\tau)y_n(\tau)=0. \quad (35)$$

Linearly independent solutions (35) have the form [9, 12]:

$$y_{1n}(\tau)=a_n(\varepsilon\tau)\sin w_n(\tau); \quad (36)$$

$$y_{2n}(\tau)=a_n(\varepsilon\tau)\cos w_n(\tau). \quad (37)$$

Here the functions $a_n(\varepsilon\tau)$ and $w_n(\tau)$ are determined up to values of ε^2 the order of from the following system of equations obtained using the asymptotic method considered in [12]:

$$\begin{cases} \dfrac{dw_n(\tau)}{d\tau} = \omega_{0n}(\varepsilon\tau); \\ \dfrac{da_n(\varepsilon\tau)}{d\tau} = -\dfrac{a_n(\varepsilon\tau)}{2\omega_{0n}(\varepsilon\tau)} \cdot \dfrac{d\omega_{0n}(\varepsilon\tau)}{d\tau}. \end{cases}$$

The solution of this system, up to a constant, has the form

$$w_n(\tau) = \int_0^\tau \omega_{0n}(\varepsilon\tau)d\tau; \qquad (38)$$

$$a_n(\varepsilon\tau) = \frac{1}{\sqrt{\omega_{0n}(\varepsilon\tau)}}. \qquad (39)$$

As $\omega_{0n}(\varepsilon\tau) > 0$, then it follows from equalities (38), (39) that $w_n(\tau)$ – monotonically increasing function.

Returning to solution (34), taking into account (36), (37), we get:

$$y_n(\tau) = a_n(\varepsilon\tau)\sin w_n(\tau)\int_o^\tau \frac{M_n(\varepsilon\zeta)\cos W_n(\zeta)\cos w_n(\zeta)}{a_n(\varepsilon\zeta)w_n'(\zeta)}d\zeta -$$
$$-a_n(\varepsilon\tau)\cos w_n(\tau)\int_o^\tau \frac{M_n(\varepsilon\zeta)\cos W_n(\zeta)\sin w_n(\zeta)}{a_n(\varepsilon\zeta)w_n'(\zeta)}d\zeta.$$

Using the well-known trigonometric formulas taking into account (25), we obtain the following expression for the total amplitude of oscillations corresponding to the nth dynamic mode:

$$A_n^2(\tau) = \frac{1}{4}A_{0n}^2(\varepsilon\tau)a_n^2(\varepsilon\tau)\left\{\left[\int_0^\tau F_n(\varepsilon\zeta)\cos\Phi_{n1}(\zeta)d\zeta + \int_0^\tau F_n(\varepsilon\zeta)\cos\Phi_{n2}(\zeta)d\zeta\right]^2 + \right.$$
$$\left. + \left[\int_0^\tau F_n(\varepsilon\zeta)\sin\Phi_{n1}(\zeta)d\zeta + \int_0^\tau F_n(\varepsilon\zeta)\sin\Phi_{n2}(\zeta)d\zeta\right]^2\right\},$$

where

$$a_n(\varepsilon\tau) = \frac{1}{\sqrt{\omega_{0n}(\varepsilon\tau)}}; \; F_n(\varepsilon\zeta) = \frac{M_n(\varepsilon\zeta)}{a_n(\varepsilon\zeta)w_n'(\zeta)}; \; w_n(\tau) = \int_0^\tau \omega_{0n}(\varepsilon\tau)d\tau;$$

$$M_{nji}(\varepsilon\tau) = \frac{-B_{ji}\omega_{0n}^2(\varepsilon\tau)Q_{nji}(\varepsilon\tau)}{A_{0n}(\varepsilon\tau)}; \; \Phi_n(\zeta) = w_n(\zeta) - W_n(\zeta).$$

Integrals containing $\sin \$_{n1}(\zeta)$, $\cos \$_{n1}(\zeta)$, increase during the entire period while the resonance phenomenon is observed, and make the main contribution to the amplitude. Ignoring members containing $\Phi_{n2}(\zeta)$, we obtain the following expression for the vibration amplitude:

$$A_n^2(\tau) = \frac{1}{4}A_{0n}^2(\varepsilon\tau)a_n^2(\varepsilon\tau)\left\{\left[\int_0^\tau F_n(\varepsilon\zeta)\cos\Phi_{n1}(\zeta)d\zeta\right]^2 + \left[\int_0^\tau F_n(\varepsilon\zeta)\sin\Phi_{n1}(\zeta)d\zeta\right]^2\right\} \qquad (41)$$

Some Results of Numerical Studies of Lateral Vibrations of a Rope of a Handling Unit

As an example, consider the transverse vibrations of a rope of a lifting installation, one end of which is wound on a drum, and a load is hinged on the other. With the help of the given model, it is possible to calculate the resonance properties of the bearing links of a wide range of lifting machines.

The equation that takes into account the bending stiffness and tension of the vibrating link has the form [9]

$$U_{tt}(x,t) + \frac{EI}{\rho}U_{xxxx}(x,t) - a^2U_{xx}(x,t) = 0. \qquad (42)$$

Border conditions

$$U(0,t) = 0; \; U_{xx}(0,t) = 0; \qquad (43)$$

$$U(l_0(t),t) = B\cos W_0(\omega_0 t); \; U_x(l_0(t),t) = 0. \qquad (44)$$

In problem (42) - (44), the following designations are used: $U(x,t)$ – lateral displacement of the link point with coordinate *x* at time *t*; *I* – axial moment of inertia of the rope section; ρ – linear mass density; $a = \sqrt{T/\rho}$ – the minimum speed of wave propagation; *T* – is the tensile force; $l_0(t) = L_0 - v_0 t$ – the law of movement of the border of the rope, L_0 – the initial length of the rope, v_0 – the speed of movement of the border; $W_0(z)$ – class function C^2; B, ω_0 – constant values; *E* – is the modulus of elasticity of the rope material.

Introduce dimensionless variables into problem (42) - (44):

$$\xi = \omega_0 x/a; \; \tau = \omega_0 t + \frac{\omega_0 L_0 - a}{-v_0}; \; U(x,t) = BV(\xi,\tau).$$

Then the problem will take the form

$$V_{\tau\tau}(\xi,\tau) + \beta^2 V_{\xi\xi\xi\xi}(\xi,\tau) - V_{\xi\xi}(\xi,\tau) = 0; \qquad (45)$$

$$V(0,\tau) = 0; \; V_{\xi\xi}(0,\tau) = 0; \qquad (46)$$

$$V\big(l(\varepsilon\tau),\tau\big) = \cos W(\tau); \; V_\xi\big(l(\varepsilon\tau),\tau\big) = 0, \qquad (47)$$

where

$$\beta^2 = \frac{EI}{\rho}\frac{\omega_0^2}{a^4}; \; l(\varepsilon\tau) = 1 + \varepsilon\tau; \; W(\tau) = W_0(\tau - \gamma_0);$$

$$\gamma_0 = \frac{\omega_0 L_0 - a}{-v_0}; \; \varepsilon = -v_0/a.$$

Note that the value of the quantity β in technical problems usually does not exceed 0,25.

Integrating equation (45) over ξ and getting rid of inhomogeneities in the boundary conditions by analogy with (6) - (7), we obtain an integro-differential equation for transverse vibrations of a rope of variable length in the form:

$$V(\xi,\tau) = -\int_0^{l(\varepsilon\tau)} K(\xi,\zeta,\varepsilon\tau)\left[V_{\tau\tau}(\zeta,\tau) + H_{\tau\tau}(\zeta,\tau)\right]d\zeta. \qquad (48)$$

The kernel of equation (48) in the case under consideration will be determined by the function

$$K(\xi,\zeta,\varepsilon\tau) = \begin{cases} \left(\dfrac{l(\varepsilon\tau)-\xi}{\beta}\right)^2\left(\dfrac{l(\varepsilon\tau)-\xi}{3} + \dfrac{\xi-\zeta}{2}\right), \zeta \le \xi, \\ \left(\dfrac{l(\varepsilon\tau)-\zeta}{\beta}\right)^2\left(\dfrac{l(\varepsilon\tau)-\zeta}{3} + \dfrac{\zeta-\xi}{2}\right), \zeta \ge \xi. \end{cases} \qquad (49)$$

Function (49) is also symmetric with respect to arguments ξ and ζ and depends on time through the parameter contained in it $\varepsilon\tau$. When fixed $l(\varepsilon\tau) = const$ function (49) coincides with the function of the influence of deflections of a rope of constant length.

Thus, problem (45) - (47) is reduced to an integro-differential equation (48) with a symmetric time-varying kernel (49) and time-varying integration limits.

The solution of problem (48) will be carried out in dimensionless variables in accordance with the method described above.

As a result, for the amplitude of oscillations corresponding to the nth dynamic mode, we obtain the following expression:

$$A_n^2(\tau)=E_n^2(\varepsilon\tau)\left\{\left[\int_0^\tau F_n(\varepsilon\zeta)\cos\Phi_n(\zeta)d\zeta\right]^2+\left[\int_0^\tau F_n(\varepsilon\zeta)\sin\Phi_n(\zeta)d\zeta\right]^2\right\},$$

where

$$E_n^2(\varepsilon\tau)=\frac{1}{4A_{1n}(\varepsilon\tau)\omega_{0n}(\varepsilon\tau)};\ \ \Phi_n(\zeta)=w_n(\zeta)-W_n(\zeta);$$

$$F_n(\varepsilon\zeta)=Q_{n_{21}}(\varepsilon\zeta)\sqrt{\omega_{0n}^3(\varepsilon\zeta)A_{1n}(\varepsilon\zeta)}.$$

The phenomenon of steady-state resonance in the system under consideration is observed if

$$W_n(\tau)=w_n(\tau)+\gamma,$$

where γ – is a constant.

Under the action of a harmonic perturbation on the system with a frequency ω_0, when $W(\tau)=\tau$, on any of the dynamic modes, the phenomenon of passage through a resonance may occur. The point of the resonance region τ_0, in which $\Phi_n'(\tau_0)=0$, is approximately determined by the following formula:

$$\tau_0=\frac{1}{\varepsilon}\left[\sqrt{\frac{2\beta^2}{-1+\sqrt{1+4\beta^2}}}\cdot\pi n-1\right].$$

To study the phenomenon of passage through resonance, it is necessary to find the values of τ_1 and τ_2 at which the square of the amplitude

$$A_n^2(\tau_1,\tau_2)=E_n^2(\varepsilon\tau_2)\left\{\left[\int_{\tau_1}^{\tau_2} F_n(\varepsilon\zeta)\cos\Phi_n(\zeta)d\zeta\right]^2+\left[\int_{\tau_1}^{\tau_2} F_n(\varepsilon\zeta)\sin\Phi_n(\zeta)d\zeta\right]^2\right\} \quad (50)$$

has a maximum.

Using the TB-Analisys[2] software complex, developed in the Matlab environment, the dependence of the maximum amplitude of the transverse vibrations of the rope when passing through resonance in the first and second dynamic modes on the relative velocity of the boundary movement at various values of the dimensionless coefficient characterizing the bending stiffness of the object was established (Table 1).

Table 1. **Dependence of the vibration amplitude A_n on ε and β when passing through resonance in the first and second dynamic modes**

	$\beta \backslash \varepsilon$	0,02	0,04	0,06	0,08
1 mode	0,01	17,3	10,7	8,8	6,7
	0,2	14,1	9,2	7,3	5,4
2 mode	0,01	12,5	7,7	5,1	4,2
	0,2	9,3	5,4	4,3	3,7

Analysis of the results obtained allows us to draw the following conclusions:

– with a decrease ε the amplitude of the oscillations increases;

– when $\varepsilon \to 0$ the amplitude of oscillations tends to infinity;

– with an increase in the mode number and bending stiffness of the object, the maximum vibration amplitude decreases.

Conclusion

An approximate method for constructing solutions of integro-differential equations is extended to a wider class of model boundary value problems on the vibrations of objects with moving boundaries in a linear formulation described by equations of hyperbolic type. This method allows taking into account the effect on the system of resistance forces of the external environment, bending stiffness and rigidity of the object substrate. The solution of the problem is brought to obtain the quadrature formulas of the amplitude of oscillations corresponding to the *n*-th dynamic mode. The phenomenon of steady-state resonance and passage through resonance is investigated by numerical methods for transverse vibrations of a rope of a hoisting machine. The above results allow at the design stage to prevent the possibility of high-amplitude oscillations in mechanical objects with moving boundaries. Using the TB-Analisys software complex, developed in the Matlab environment, the dependence of the maximum amplitude of the transverse vibrations of the rope when passing through resonance in the first and second dynamic modes on the relative velocity of the boundary movement at various values of the dimensionless coefficient characterizing the bending stiffness of the object was established.

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